

ON EXCEPTIONAL ENRIQUES SURFACES

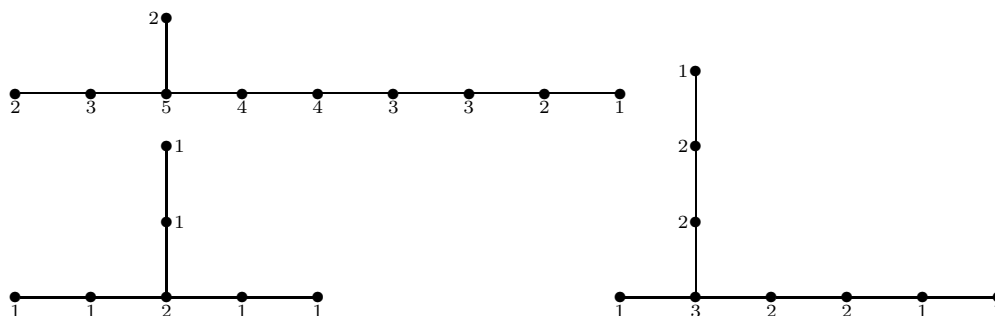
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ABSTRACT. We give a complete description of all classical (“ $\mathbf{Z}/2$ ”) Enriques surfaces with non-zero global vector fields. In particular we show that there are such surfaces. The obtained result also applies to supersingular (“ α_2 ”) Enriques surfaces fulfilling a rather special condition. During this classification we study some properties of genus 1-fibrations special to characteristic as well as make a close study of the genus 1-fibration on the surfaces that we classify.

Whether or not a classical Enriques surface may have non-zero global vector fields is a question of great interest to for instance the deformation theory of Enriques surfaces. In [SB96] it was claimed that this may never happen. There is however an error in the proof and we shall see that the truth is the opposite; there does indeed exist classical Enriques surfaces with non-zero global vector fields. It turns out that a condition that in the case of a classical Enriques surface is equivalent to having a non-zero global vector field is of interest also in the non-classical case; we shall call surfaces fulfilling that condition *exceptional* (see (0.3) for the precise definition). Our main result is formulated in the following Theorem. In it the conductrix is a specific divisor whose support is the image of divisorial part of the singular locus of the canonical double cover. We shall say that an exceptional surface is of *type* Γ if the support of the conductrix forms a Γ -configuration. (For the precise definition of the conductrix as well as the definition of the graphs $T_{p,q,r}$ and the notion of Γ -configuration, see the preliminaries). Let us also agree to say that a genus 1 fibration on an Enriques surface is *special* if it has a 2-section of (arithmetic) genus 0 (it always has a 2-section of genus 0 or 1). A 2-section of genus 0 will also be called special. (Classically surfaces with such a pencil are called *special*. It is called a *degenerate U-pair* in [CD89].) Such a 2-section will also be referred to as a *special 2-section*.

Theorem A *Let X be an Enriques surface in characteristic 2.*

i) X is exceptional precisely when its conductrix forms one of the following configurations with the indicated multiplicities.



In particular, an exceptional surface is of type $T_{3,3,3}$, $T_{2,4,5}$, or $T_{2,3,7}$.

ii) X is exceptional if and only if it admits a special genus 1 fibration with a double fibre of type \tilde{E}_6 , \tilde{E}_7 , or \tilde{E}_8 . It is then of type $T_{3,3,3}$, $T_{2,4,5}$, or $T_{2,3,7}$ respectively.

iii) X is exceptional if it admits one of the following genus 1-fibrations:

- A quasi-elliptic fibration with a simple \tilde{E}_7 -fibre, X is then of type $T_{3,3,3}$.*
- A quasi-elliptic fibration with a simple \tilde{E}_8 -fibre, X is then of type $T_{2,4,5}$.*

The main part of the argument consists of a rather detailed study of what happens to fibres of genus 1-fibrations under pullback by the Frobenius map on the base of the fibration (in characteristic 2) which may be of independent interest. The definition of exceptional Enriques surface is given by a simple condition on the conductrix but we also give the following elaboration of that condition.

Theorem B *An exceptional Enriques surface is either a $\mathbf{Z}/2$ - (“classical”) or an α_2 -surface (“supersingular”). A $\mathbf{Z}/2$ -surface is exceptional precisely when it has global vector fields and then the dimension of the space of global vector fields is 1. An α_2 -surface is exceptional precisely when the cup products of all elements of $H^1(X, \mathcal{O}_X)$ and all elements of $H^0(X, \Omega_X^1)$ are zero. Both cases occur.*

Remark: The presence of vector fields on a $\mathbf{Z}/2$ -surface is clearly making its deformation theory “pathological”. We shall show elsewhere that an Enriques surface is exceptional precisely when a versal deformation of it as unipotent Enriques surface (a notion that in the point case is equivalent to being a $\mathbf{Z}/2$ - or α_2 -surface) is singular.

We then go on to discuss the classification of exceptional surfaces (and show in particular that all three types exist for $\mathbf{Z}/2$ -surfaces as well as for α_2 -surfaces) and give a description of all genus 1-fibrations on them. This description is somewhat complicated in the case of exceptional surfaces of type $T_{3,3,3}$; in that case we need to distinguish between surfaces of different *MW-rank*, which by definition equals $8 - \sum_s n(s) - 1$, where s runs over the fibres of the unique elliptic pencil on X and $n(s)$ is the number of irreducible components of s . (The MW-rank is also the rank of the Mordell-Weil group of the Jacobian of the generic fibre of the pencil, hence justifying the name.)

Theorem C *i) An exceptional Enriques surface of type $T_{2,3,7}$ has a unique genus 1-fibration which is quasi-elliptic.*

ii) An exceptional Enriques surface of type $T_{2,4,5}$ has two or three genus 1-fibrations all of which are quasi-elliptic.

iii) An exceptional Enriques surface of type $T_{3,3,3}$ has a unique elliptic fibration. There are quasi-elliptic fibrations, which are arranged in triples; the set of triples is a torsor under a discrete group, which is trivial if the MW-rank is zero, \mathbf{Z} if the MW-rank is 1, and the Coxeter group of A_2 if the MW-rank is 2. Each quasi-elliptic fibration appears in 1 or 2 of these triples.

Note that we also give a description (see Theorem 4.7) of the -2 -curves on an exceptional surface.

Remark: The proofs of Theorems A and B will be found on page 23 after the proof of Lemma 4.1 and the proof of Theorem C on page 32 after Proposition 4.6.

A substantial part of this paper consists of a somewhat tedious enumeration of the possibilities for various integer weightings of Dynkin diagrams. It would certainly be possible (and was at one point done by us in the quasi-elliptic case) to cut down on the size of the proofs by performing these enumerations mechanically on a computer. However, we feel that the current proofs and their use of the notion of admissible weightings gives a rather strong indication as to why the list of possible weightings is as small as it actually is; as the proof of Proposition 1.5 shows, the admissible weightings fulfill a rather strong extremality condition. A mechanical enumeration on the other hand gives no such indication. (Also the source code for such an enumeration would be rather long – though only repetitions with small modifications of a rather short template – and hence error prone.)

Conventions: To simplify announcements we shall assume that the base field of all our varieties is, unless explicitly claimed otherwise, algebraically closed of characteristic 2.

We shall name the types of Enriques surfaces in characteristic 2 after the type of their Pic^T , the correspondence with another established terminology is that μ_2 -surfaces are also called *singular*, $\mathbf{Z}/2$ -surfaces *classical* and α_2 -surfaces *supersingular*. In this article we shall exclusively be interested in the α_2 - and $\mathbf{Z}/2$ -case and shall refer to such a surface as a *unipotent* surface.

We shall use the extended Dynkin diagram notation for the fibres of a minimal genus fibration (over a 1-dimensional base): If the components of the fibre form a normal crossing divisor we use the diagram that is the dual graph of the fibre. When the fibre is irreducible we denote it by \tilde{A}_0 , \tilde{A}_0^* , and \tilde{A}_0^{**} as it is smooth, nodal, or cuspidal respectively. When it consists of two non-transversal components we use \tilde{A}_1^* and if it has three components meeting in one point we denote it by \tilde{A}_2^* . In the \tilde{A}_2^* case we shall by a slight abuse of language speak of \tilde{A}_2 as the dual graph of the fibre and we shall say that in that case, as in the case when the fibre is a normal crossing divisor that the fibre *has a dual graph*. It is clear that when the fibre has a dual graph, the intersection matrix of the components of the fibre is indeed described by its dual graph. Note also that starting after Proposition 0.4 an Enriques surface will, unless otherwise mentioned, be assumed to be unipotent.

The E -series of (extended) Dynkin diagrams are also graphs of type $T_{*,*,*}$ (cf. [CD89, p. 105]) and we shall freely pass back and forth between the two notations.

Preliminaries

Our first preliminary result will be stated in far greater generality than will actually be needed.

Lemma 0.1 *Let $\pi: X \rightarrow S$ be a proper map of relative dimension $\leq n$ with S an affine noetherian scheme. Suppose that $H^n(X, \mathcal{O}_X) \neq 0$ and that this is false for any proper closed subscheme of X . Then*

- i) $H^0(X, \mathcal{O}_X)$ is a field and
- ii) \mathcal{O}_X contains no non-zero subsheaves of support of relative dimension $< n$.

PROOF: By assumption $H^n(X, -)$ is right exact on quasi-coherent sheaves. Let $0 \neq \lambda \in R := H^0(X, \mathcal{O}_X)$ and let X_λ be the closed sub-scheme defined by λ so that we have an exact sequence $\mathcal{O}_X \xrightarrow{\lambda} \mathcal{O}_X \rightarrow \mathcal{O}_{X_\lambda} \rightarrow 0$. By right exactness we get an exact sequence $H^n(X, \mathcal{O}_X) \xrightarrow{\lambda} H^n(X, \mathcal{O}_X) \rightarrow H^n(X, \mathcal{O}_{X_\lambda}) \rightarrow 0$. As X_λ is a proper subscheme we have by assumption that $H^n(X, \mathcal{O}_{X_\lambda}) = 0$ and so multiplication by λ is surjective. Now as $H^n(X, \mathcal{O}_X)$ is finitely generated over $\Gamma(S, \mathcal{O}_S)$ and thus finitely generated over R and non-zero, it has a non-zero quotient killed by some maximal ideal m of R . Multiplication by any non-zero $\lambda \in m$ would then be surjective as well as zero on this quotient. Hence $m = 0$ and we get that R is a field. As for ii) the closed subscheme of X defined by such a subsheaf has the same $H^n(-, \mathcal{O}_-)$ as X does and hence the subsheaf is zero by assumption. \square

We shall need the following quite specialised result in order to relate the existence of non-zero global vector fields on a $\mathbf{Z}/2$ -surface to genus 1-fibrations.

Lemma 0.2 *Let D be an effective divisor on an Enriques surface X for which $h^0(X, \mathcal{O}(D)) = 1$ and $h^1(\mathcal{O}_D) \neq 0$. Then D contains a half-fibre of a genus 1 fibration.*

PROOF: We begin by noting that, by Riemann-Roch and the assumption $h^0(X, \mathcal{O}(D)) = 1$, D contains no subdivisor of strictly positive self-intersection. We start by proving that D contains a half-fibre. For this it is enough to prove that D contains a subdivisor of self-intersection 0 as then by, e.g., [CD89, Thm. 3.2.1] it contains a half-fibre or a fibre but a fibre is excluded by the condition $h^0(X, \mathcal{O}(D)) = 1$.

By lemma 0.1 and noetherianity there exists an effective divisor $E \subseteq D$ which is minimal for the condition that $h^1(\mathcal{O}_-) \neq 0$ and we have, again by the lemma, that $h^0(\mathcal{O}_E) = 1$. This gives $\chi(\mathcal{O}_E) \leq 0$ and thus, by Riemann-Roch, that $E^2 \geq 0$ and as strictly positive self-intersection was impossible we conclude. \square

We shall be interested in a particular divisor on an Enriques surface to which we shall have occasion to apply the previous lemma. We recall that if S is a Gorenstein scheme, X and Y degree 2 flat S -schemes and $\pi: Y \rightarrow X$ a finite birational S -map then to begin with, and irrespective of S , we have the conductor ideal $\mathcal{I}_C \subseteq \mathcal{O}_X$ of π which is the maximal ideal in \mathcal{O}_X which is

also an \mathcal{O}_Y -ideal. As X and Y are Gorenstein schemes, \mathcal{I}_C is an invertible ideal. However, under the stated assumptions there is a canonical effective Cartier divisor A on S such that the \mathcal{I}_C is the pullback of $-A$. In fact we have exact sequences $0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_X \rightarrow L \rightarrow 0$ and $0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_Y \rightarrow L' \rightarrow 0$ where L and L' are line bundles and π induces an injective map $\mathcal{O}_X \rightarrow \mathcal{O}_Y$ which in turn induces an injective map $L \rightarrow L'$ giving a Cartier divisor on \mathcal{O}_S . The proof follows from standard duality theory using that $\omega_{X/S} = \mathcal{O}_X \otimes L^{-1}$ and $\omega_{Y/S} = \mathcal{O}_Y \otimes L'^{-1}$. We shall call this Cartier divisor the *conductrix* of the S -map π .

In the particular situation when S is an Enriques surface, X its canonical double cover, and Y its normalisation we shall, by a small abuse of language, speak of the conductrix of S . Twice the conductrix will also play an important rôle in what is to follow and we shall call it the *bi-conductrix*.

If instead we have a genus 1 fibration $\pi: X \rightarrow S$ in characteristic 2 with S 1-dimensional and X and S regular then we get a map $\pi': X' \rightarrow S$ which is the pullback of π by the Frobenius map on S and the normalisation map $\rho: \tilde{X} \rightarrow X'$. The conductrix of π is then by definition the conductrix of ρ . This leads to a small ambiguity for a genus 1 pencil on an Enriques surface as we have then the conductrix of the surface and of the pencil. This should cause no confusion however (and is no formal ambiguity as the two conductrices are associated to two different objects).

In the Enriques surface case the following somewhat mysterious-looking condition on the bi-conductrix will be of great importance in what will follow.

Definition 0.3 *An Enriques surface will be called exceptional if $H^1(B, \mathcal{O}_B) \neq 0$ where B is the bi-conductrix.*

Our first result excludes “most” Enriques surfaces from being exceptional.

Proposition 0.4 *A non-unipotent Enriques surface (in particular all such surfaces in characteristic different from 2) has empty conductrix. In particular it is not exceptional.*

PROOF: When the canonical double cover is étale the double cover is normal and hence the conductrix is empty. This is the case (precisely) when the surface is not unipotent. \square

In view of this proposition we shall from now make the blanket assumption that *unless otherwise mentioned all our Enriques surfaces are unipotent*.

We now collect some properties of the conductrix including an unfolding of the meaning of exceptionality in the $\mathbf{Z}/2$ -case.

Proposition 0.5 *Let X be an Enriques surface, A its conductrix, and B its bi-conductrix.*

- i) *The bi-conductrix is the divisorial part of the zero-set of any non-zero global 1-form.*
- ii) *We have $h^0(\mathcal{O}_X(B)) = 1$. In particular the conductrix can not contain a fibre or half-fibre of a genus 1-fibration.*
- iii) *Assume that the conductrix is non-zero. Then it is 1-connected and all effective subdivisors of it have strictly negative self-intersection. Furthermore, $A^2 = -2$ and the normalisation of the canonical double cover has either 4 ordinary double points as singularities or one rational double point of type D_4 .*
- iv) *The minimal resolution of the normalisation of the canonical double cover has $h^{01} = 0$.*
- v) *X is exceptional iff B contains a half fibre of some genus 1-fibration.*
- vi) *If X is a $\mathbf{Z}/2$ -surface then X is exceptional if and only if it has a non-trivial global vector field. In any case $h^0(X, T_X) \leq 1$.*
- vii) *If X is an α_2 -surface then X is exceptional precisely when the cup product*

$$H^1(X, \mathcal{O}_X) \otimes H^0(X, \Omega_X^1) \rightarrow H^1(X, \Omega_X^1)$$

is zero.

PROOF: Consider the universal $(\underline{Pic}^\tau)^\vee$ -torsor $Z \rightarrow X$ and its normalisation $\tilde{Z} \rightarrow Z$. Then we have an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \pi_* \mathcal{O}_{\tilde{Z}} \rightarrow \omega_X(A) \rightarrow 0,$$

where $\pi: \tilde{Z} \rightarrow Z$ is the structure map. Recall that the map $f \mapsto df^2$ from $\pi_*\mathcal{O}_{\tilde{Z}}$ to \mathcal{O}_X induces an injective map $\mathcal{O}_X(B) \cong F^*(\omega_X(A)) \rightarrow \Omega_X^1$ and it is saturated as \tilde{Z} is normal. This shows that B is the divisorial part of the zero-set of a 1-form and as $h^0(\Omega_X^1) = 1$ the same is true for any non-zero 1-form. We also get an exact sequence

$$0 \rightarrow \mathcal{O}_X(B) \rightarrow \Omega_X^1 \rightarrow \mathcal{I}_W\omega_X(-B) \rightarrow 0, \quad (0.6)$$

where W is a zero-dimensional subscheme which is non-empty if $B = 0$ as is seen by computing Chern classes. From the fact that $h^0(\Omega_X^1) = 1$ we then get $h^0(\mathcal{O}_X(B)) = 1$.

By [CD89, Prop. 3.1.2, Thm. 3.2.1] any effective divisor C with $C^2 \geq 0$ has $h^0(\mathcal{O}(2C)) \geq 2$. Hence, we have $A^2 < 0$ if $A \neq 0$. On the other hand, if $A > 0$ then \tilde{Z} is rational and if $A = 0$ then already Z is normal and $h^1(\mathcal{O}_Z) = 0$ and $h^2(\mathcal{O}_Z) = 1$. From the classification of surfaces and the fact that $\chi(\mathcal{O}_-)$ decreases under resolution of singularities we get that $\chi(\mathcal{O}_{\tilde{Z}})$ is 1 or 2. As also $\chi(\mathcal{O}_{\tilde{Z}}) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}(A)) = A^2/2 + 2$ we see that $A^2 \geq -2$ and hence either $A = 0$ or $A^2 = -2$. As $h^1(\mathcal{O})$ for a normal surface can only increase under desingularisation we see that $h^1(\mathcal{O}_Y) = 0$ if it is rational. In the other case $A = 0$ and then we have already seen that $h^1(\mathcal{O}_Y) = 0$. Finally assume that A can be written as a sum $C + D$ of effective non-zero divisors. We have that $C^2, D^2 < 0$ as otherwise $2C$ or $2D$ would move and so would B and thus $C^2, D^2 \leq -2$ as they are even. Hence, $-2 = A^2 = C^2 + 2C \cdot D + D^2 \leq -4 + 2C \cdot D$ which gives $C \cdot D > 0$.

As we know that $A^2 = -2$ we may compute the order of the zero locus of the characteristic map $\mathcal{O}_X(2A) \rightarrow \Omega_X^1$ using Chern classes (and the fact that the zero set is isolated) and the result is $12 + (2A)^2 = 4$. Now, at a singular point of the \tilde{Z} we may write the completion of the local ring as $\mathbf{k}[[x, y, z]]/(z^2 - f(x, y))$ with $f \in m_{(x, y)}^2$ and the order of the zero locus at that point is the dimension of $\mathbf{k}[[x, y]]/(f'_x, f'_y)$. A simple calculation shows that either the quadratic part of f contains xy in which case the singularity is an ordinary double point or the cubic part is a square free cubic polynomial in which case the singularity is a D_4 -singularity or the dimension is strictly greater than 4. As the order of the zero locus is the sum of the local orders at singular points we get either four ordinary double points or one D_4 -point.

As for v) any effective subdivisor D' of an effective divisor D on X with $h^1(\mathcal{O}_D) = 0$ has $h^1(\mathcal{O}_{D'}) = 0$ by right exactness of $H^1(D, -)$ on quasi-coherent sheaves which gives one direction. The other direction follows from lemma 0.2.

To continue we notice that by duality, $H^1(B, \mathcal{O}_B) \neq 0$ is equivalent with $H^0(B, \omega_B) \neq 0$. Furthermore we have the standard exact sequence

$$0 \rightarrow \omega_X \rightarrow \omega_X(B) \rightarrow \omega_B \rightarrow 0. \quad (0.7)$$

In the $\mathbf{Z}/2$ -case we exploit the short exact sequence

$$0 \rightarrow \omega_X(B) \rightarrow T_X \rightarrow \mathcal{I}_W\mathcal{O}(-B) \rightarrow 0,$$

which is dual to (0.6). From it it follows that if $H^0(T_X) \neq 0$ then $H^0(\omega_X(B)) \neq 0$ but $H^0(X, \omega_X) = 0$ and we conclude by the long exact sequence associated to (0.7). This proves vi).

As for vii) assume X is an α_2 -surface. We start by noticing that as $H^1(X, \mathcal{O}_X)$ and $H^0(X, \Omega_X^1)$ are 1-dimensional, the cup product is zero precisely when the cup product of two non-zero elements is. Thus let β be a non-zero element of $H^1(X, \mathcal{O}_X)$ and η a non-zero element of $H^0(X, \Omega_X^1)$. By (0.6) and the fact that $W \neq 0$, η is the image of some $\eta' \in H^0(X, \mathcal{O}(B))$ and thus $\eta\beta$ is the image of $\eta'\beta$. Now, again by (0.6) and the fact that $W \neq 0$ we get that $H^1(X, \mathcal{O}(B)) \rightarrow H^1(X, \Omega_X^1)$ is injective so that $\eta\beta$ is zero precisely when $\eta'\beta$ is. Now as we have just showed that $h^0(\mathcal{O}_X(B)) = 1$, we may assume that η' in turn comes from $1 \in H^0(X, \mathcal{O}_X)$ under the inclusion of (0.7). Thus $\eta'\beta$ is the image of β under the map $H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}(B))$. As ω_X is trivial (X being an α_2 -surface), the inclusion map $\mathcal{O}_X \hookrightarrow \mathcal{O}(B)$ is isomorphic to the inclusion $\omega_X \hookrightarrow \omega(B)$. It follows from (0.7) and that $h^0(\omega_X) = h^0(\omega(B)) = 1$, that the map

$H^1(X, \omega_X) \rightarrow H^1(X, \omega_X(B))$ is zero precisely when $h^0(\omega_B) \neq 0$, which as we have noticed is equivalent to X being exceptional. \square

We need to keep careful track of what happens with -2 -curves on an Enriques surface when taking their inverse images in the normalisation of the canonical double cover. However, we shall also want to work with a genus 1-fibration over a discrete valuation ring so that we shall work with general surfaces by which we shall mean a 2-dimensional Noetherian scheme. Recall that for a double cover $\pi: Y \rightarrow X$ of surfaces, with X regular and Y normal a resolution of Y may be obtained by successively blowing up X in the points below singular points of Y and taking its normalisation in $\mathbf{k}(Y)$. This blowing up of X , which is uniquely determined by π , will be referred to as the *minimal dissolution* of π . When Y has only rational singularities, the result is the minimal resolution of Y (but in general the resolution may not be minimal). By definition a -2 -curve on X will be a smooth proper genus 0-curve with self-intersection -2 (and hence with zero intersection with the canonical divisor).

Definition-Lemma 0.8 *Let X be a regular surface in characteristic 2 and $\pi: Y \rightarrow X$ an inseparable flat double cover with Y normal and $\rho: \tilde{Y} \rightarrow Y$ the normalisation of the minimal dissolution of π . Let A be the line bundle $(\pi_* \mathcal{O}_Y / \mathcal{O}_X)^{-1}$ of π and E a -2 -curve on X . We associate the following invariants to E , where \tilde{E} is the irreducible curve in \tilde{X} mapping surjectively to E , which we shall the strict inverse image.*

1. The degree s of $\rho \circ \pi: \tilde{E} \rightarrow E$. This number is 1 or 2.
 2. The number r of points (including infinitely close points) on E that are blown up during the minimal dissolution of π .
 3. The intersection number $A \cdot E$.
 4. The self intersection \tilde{E}^2 .
 5. The genus g of \tilde{E} .
- i) We have the relations $\tilde{E}^2 = (-2 - r)s^2/2$ and $2g - 2 = (-2 - r)s^2/2 - sA \cdot E$.
ii) When $(A, E) \geq -2$ the possible values for these invariants are given by the following table

| r | s | (A, E) | \tilde{E}^2 | g |
|-----|-----|----------|---------------|-----|
| 0 | 1 | 1 | -1 | 0 |
| 0 | 2 | -1 | -4 | 0 |
| 2 | 1 | 0 | -2 | 0 |
| 4 | 1 | -1 | -3 | 0 |
| 6 | 1 | -2 | -2 | 0 |
| 1 | 2 | -2 | -6 | 0 |

We shall say that E is of self-intersection type \tilde{E}^2 .

iii) If two -2 -curves on X meet transversally then either their s -invariants are different, $s = 2$ for both of them and their strict inverse images on \tilde{X} meet non-transversally, or $s = 1$ for both of them and their strict transforms do not meet on the minimal dissolution of π . In the last case, their r -invariants are positive.

PROOF: The relation $\tilde{E}^2 = (-2 - r)s^2/2$ follows directly from the fact that the self-intersection goes down by one when a point on a curve is blown up and the fact that $2/s\tilde{E} = \pi'^{-1}E'$, where E' is the strict transform of E on the minimal dissolution and π' is the map to X from \tilde{X} . The second relation follows from the first, the adjunction and projection formulas and the fact that $\omega_{Y/X} = \pi^*(\mathcal{O}_X(-A))$.

As for the table we may certainly assume that $A \neq 0$ as it is obvious if E is not contained in A . Furthermore, if $s = 1$ then \tilde{E} maps birationally onto E and hence $g = 0$. Using this together

with the fact that generally $g \geq 0$ (as \tilde{E} is integral) and the formula $2g - 2 = (-2 - r)s^2/2 - A \cdot E$ the table is easily established.

If the strict transform on the minimal dissolution of the two curves are E_1 and E_2 and $\tilde{\pi}$ is the map from \tilde{X} to the minimal dissolution then we have $2(E_1 \cdot E_2) = (2/s_1)(2/s_2)(\tilde{E}_1 \cdot \tilde{E}_2)$ and as $E_1 \cdot E_2$ is 0 or 1 this immediately gives the last statement. \square

We shall need the following extension of a result of Shepherd-Barron.

Lemma 0.9 *Let X be a $\mathbf{Z}/2$ - or α_2 -Enriques surface, $\rho: \tilde{X} \rightarrow X$ its canonical double cover and $\pi: X \rightarrow \mathbf{P}^1$ a genus 1-fibration on X .*

i) ρ factors through the pullback X_F of π by the Frobenius map on \mathbf{P}^1 . The map $\tilde{X} \rightarrow X_F$ is an isomorphism outside of the double fibres of π .

ii) The restriction of ρ to a half fibre is non-trivial.

PROOF: For the first part the case of a $\mathbf{Z}/2$ -surface is [SB96, Lemma 1.7] (but we shall indicate how also that case could be treated) so we may assume that X is an α_2 -surface. We first claim that the restriction of the canonical double cover to a simple fibre F is trivial. Indeed, $H^0(F, \mathcal{O}_F)$ equals the base field \mathbf{k} so it suffices to show that the map $H^1(X, \mathcal{O}_X) \rightarrow H^1(F, \mathcal{O}_F)$ is zero but this follows immediately from the long exact sequence of cohomology associated to

$$0 \rightarrow \mathcal{O}(-F) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_F \rightarrow 0$$

and the fact that $h^1(\mathcal{O}(-F)) = 1$. (In the $\mathbf{Z}/2$ -case the triviality is even simpler as ω_X is isomorphic to the line bundle associated to the divisor that is the sum of the two half fibres minus twice a fibre which is visibly trivial when restricted to a fibre.) Consider now the affine algebra $\rho_*\mathcal{O}_{\tilde{X}}$ and its push down $\mathcal{A} := \pi_*\rho_*\mathcal{O}_{\tilde{X}}$ to \mathbf{P}^1 . As its restriction to any simple fibre is a trivial vector bundle we get that \mathcal{A} is a rank 2 vector bundle on \mathbf{P}^1 . On the other hand, the short exact sequence $0 \rightarrow \mathcal{O}_X \rightarrow \rho_*\mathcal{O}_{\tilde{X}} \rightarrow \mathcal{O}_X \rightarrow 0$ induces a long exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^1} \rightarrow \mathcal{A} \rightarrow \mathcal{O}_{\mathbf{P}^1} \rightarrow R^1\pi_*\mathcal{O}_X$$

and as \mathcal{A} has rank 2 the image of the boundary map is torsion. Now the torsion of $R^1\pi_*\mathcal{O}_X$ has length 1 so we get that $\mathcal{A}/\mathcal{O}_{\mathbf{P}^1}$ is isomorphic to $\mathcal{O}_{\mathbf{P}^1}$ or $\mathcal{O}_{\mathbf{P}^1}(-1)$. The first possibility is excluded as $H^0(\mathbf{P}^1, \mathcal{A}) = \mathbf{k}$ and thus $\mathcal{A} = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ and hence $\mathbf{Spec} \mathcal{A}$ is obtained by taking a square root of homogeneous polynomial of degree 2 (the same conclusion is easier in the $\mathbf{Z}/2$ -case as $R^1\pi_*\mathcal{O}_X$ is torsion free and $\pi_*\omega_X = \mathcal{O}_{\mathbf{P}^1}$). Up to isomorphism there are only two such covers, trivial and the Frobenius map. The trivial cover is excluded as \mathcal{A} is reduced (as \tilde{X} is) and so $\pi \circ \rho$ factors through the Frobenius map on \mathbf{P}^1 . Finally, the map $\pi^*\mathcal{A} \rightarrow \rho_*\mathcal{O}_{\tilde{X}}$ is an isomorphism outside of the double fibres.

As for the second part, the case of a $\mathbf{Z}/2$ -surface is well-known. Consider therefore an α_2 -surface and let F be the half-fibre. We consider again the long exact sequence associated to

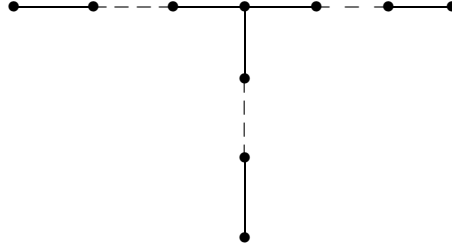
$$0 \rightarrow \mathcal{O}(-F) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_F \rightarrow 0.$$

This time $h^1(\mathcal{O}(-F)) = 0$ and so the map $H^1(X, \mathcal{O}_X) \rightarrow H^1(F, \mathcal{O}_F)$ is injective which is what is needed. \square

Definition 0.10 *Let X be a smooth and proper surface and Γ a graph. By a Γ -configuration on X we shall mean a collection of curves of genus 0 and self-intersection -2 any pair of which has intersection 0 or 1 and a bijection between its members and the vertices of Γ such that two curves intersect iff the corresponding vertices are connected in Γ .*

Finally, we recall (cf. [CD89, p. 105]) that the graph $T_{p,q,r}$ consists of a vertex of degree 3

and three arms with p , q and r vertices (including the central vertex):



1 Genus 1 fibrations and admissible weightings

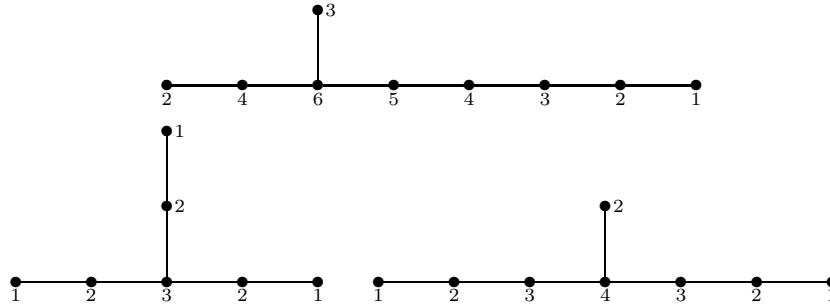
Definition 1.1 Let Γ be a simply laced extended Dynkin diagram (i.e., of type \tilde{A}_n , \tilde{D}_n , or \tilde{E}_n).

i) A vertex root of the diagram is a root associated to a vertex of Γ , these vertices forming a basis for the root lattice. (We shall follow the geometric convention for which the square of a vertex root is -2 .) An edge root of Γ is a root which is the sum of the two vertices adjacent to an edge of Γ . The Kodaira-Néron cycle is the unique positive primitive linear combination of vertex roots of square 0.

ii) If an element of the root lattice is a positive linear combination of the vertex roots yet is not a sum of the Kodaira-Néron cycle and another positive linear combination of vertex roots then the element is reduced. If an element m of the root lattice is written as a sum of a reduced element and an integral multiple of the Kodaira-Néron cycle we say that the reduced element is the fractional part of m (it is clearly unique).

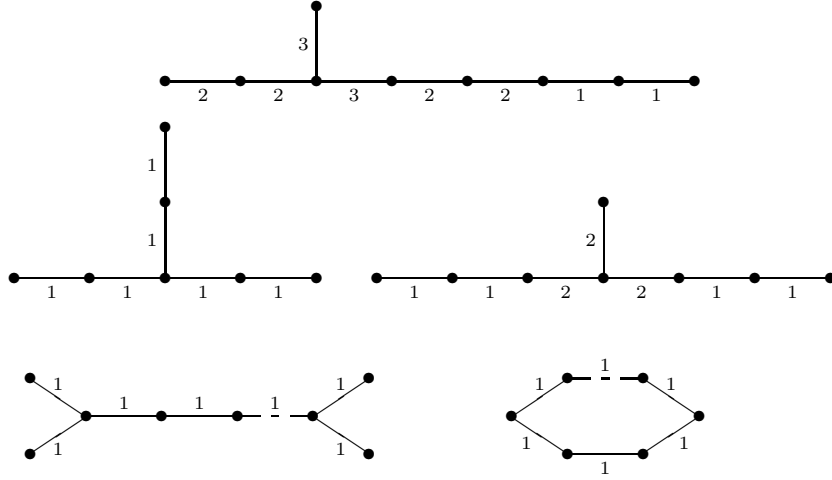
Recall, that the extended Dynkin diagram is obtained from the non-extended one by attaching one new vertex to the original diagram. We shall call this vertex the *attached vertex*. Note furthermore that the attached vertex appears with multiplicity 1 in the Kodaira-Néron cycle and that up to automorphism of the extended diagram it is the only such vertex.

As we are going to make a lot of reference to the Kodaira-Néron cycle let us recall its form: For \tilde{A}_n it is just the sum of all the vertex roots. For \tilde{D}_n the vertex roots of the vertices of degree 1 appear with multiplicity 1 and all other vertex roots with multiplicity 2. For the E series the multiplicity is given by the following diagrams:



Our first use of this information is the following lemma.

Definition-Lemma 1.2 For each simply laced extended Dynkin diagram Γ let E be the sum of the edge roots with the following multiplicities:



Then the Kodaira-Néron cycle F can be written in form $1/n(E - V)$ where $n = 1$ except in the \tilde{A}_m case where it is 2, and E is zero if Γ equals \tilde{A}_n or \tilde{E}_6 , the sum of the two degree 3 vertex roots when it equals \tilde{D}_n , $n \geq 5$ and twice the degree ≥ 3 vertex root in the \tilde{D}_4 and \tilde{E}_n , $n = 7, 8$ cases. We shall call E the excess cycle and the multiplicity with which an edge appears in it its excess multiplicity.

PROOF: A simple verification. \square

Let us consider an integral weighting (or weight) w of a simply-laced extended Dynkin diagram Γ (i.e., a function from the set of vertices to the integers). We extend the weight linearly to the root lattice. The *excess* of an edge wrt w is minus the value of w on the corresponding edge root. Minus the value of w on the excess cycle will be called the *excess* of w . If the excess of an edge is > 0 we shall say that the edge is *excessive*. Finally, the value of w on the Kodaira-Néron cycle will be called the *fibre weight* of the weight. We shall say that w is *admissible* if:

1. The weights of all the vertices is ≤ 1 .
2. The excess of all edges are ≥ 0 .
3. The fibre weight of w is 0, 1, or 2.
4. A vertex of weight 0 is adjacent to at most two other vertices of weight 0.
5. There is an element u in the root lattice such that $w(v) \geq (v, u)$ for all vertices v , where (\cdot, \cdot) is the standard (negative definite) scalar product on the root lattice. If such a u exists its fractional part will also work (as the Kodaira-Néron cycle lies in the radical) and hence u may and will be assumed to be reduced.
6. If the fibre weight of w is 2, then any vertex that appears with multiplicity 1 in the Kodaira-Néron cycle does not appear in this reduced u .

The reduced u will be called a *representing element* and the weight w' given by $w'(v) := w(v) - (v, u)$ will be called the *complement* of u (with respect to w of course). The fifth admissibility condition then forces the complement to take non-negative values on the vertex roots. For an admissible weight we call an edge of excess > 0 *excessive*. For a non-excessive edge we have either that the weights of both vertices on it are 0 in which case it will be called an *s-edge* or one vertex has weight 1 and the other weight -1 and will be called an *n-edge*. The reason for introducing the notion of admissibility is that we will get admissible weightings from genus 1-fibrations (cf. Proposition 1.3). In one case (fibres of type \tilde{A}_1^*) all but one of the conditions will be fulfilled and we say that a weight is *semi-admissible* if all the conditions of admissibility but the condition on the excess being non-negative (the second condition) is fulfilled.

The reason for our interest in admissible weights is the following result.

Proposition 1.3 *Let $\pi: X \rightarrow S$ be a genus 1 fibration over a 1-dimensional regular base (with X also regular) in characteristic 2 and let $\pi': X' \rightarrow S'$ be its pullback by the Frobenius map on S . Let $\rho: \tilde{X} \rightarrow X'$ be the normalisation map and A on X its conductrix. If s is a closed point of S , with reducible fibre not of type \tilde{A}_1^* , then the weight w on the dual graph of the fibre over s defined by $w(v) = (A, v)$ is admissible. In the case of a fibre of type \tilde{A}_1^* the weight is semi-admissible. The fibre weight is 0 if π is elliptic, 1 if π is quasi-elliptic with a simple fibre above s and 2 if π is quasi-elliptic with a double fibre above s .*

PROOF: The condition that the weight of a vertex is ≤ 1 follows from Lemma 0.8 as we have $-2 \leq 2g - 2 = (-2 - r)s^2/2 - s(A, v) < -s(A, v)$. If $(A, e) > 0$ for an edge root e then if v_1 and v_2 are the two vertices on e then for at least one of them, v_1 say, we must have $(A, v_1) > 0$. This gives that $s = 1$ and $r = 0$. As $r = 0$ we get, again from (0.8) and the fact that two curves in the fibre intersect transversally as we have excluded the \tilde{A}_1^* case, that as v_2 is adjacent to v_1 we must have that $s = 2$ for v_2 . This gives $-2 \leq 2g - 2 = (-2 - r)2 - 2(A, v_2) \leq -4 - 2(A, v_2)$ which gives $(A, v_2) \leq -1$ and thus $(A, e) \leq 1 - 1 = 0$, a contradiction. Furthermore, from (0.8) it also follows that in the minimal dissolution two points are blown up on a curve of weight 0 but also all intersection points between such curves are blown up. Hence a vertex of weight 0 can meet at most two other such vertices. Further, A may be written as a sum of effective divisors $A_s + A'$ where A_s has support in the fibre and A' has no component in the fibre. We then have $(A, v) = (A_s, v) + (A', v) \geq (A_s, v)$. More precisely, A can be decomposed as $A_s + R + A''$ where A'' has support in other fibres and R does not contain components of fibres. Then for f an element of the root lattice we have $(A, f) = (A_s, f) + (R, f)$. If f is the Kodaira-Néron cycle, then $(A_s, f) = 0$ so that $(A, f) = (R, f)$. In the elliptic case π is generically smooth and hence $R = 0$. In the quasi-elliptic case R is the curve of cusps and hence has intersection 2 with a fibre and thus (R, f) is 2 if the fibre is simple and 1 if it is double. Finally, when the fibre is simple the map is smooth at a generic point of a curve that appears with multiplicity 1 in the Kodaira-Néron cycle and hence that curve is not in the support of the conductrix. \square

The existence of a representing element requires us to be able to decide when a weight is given by scalar product by an element. We record for reference the following easy lemma.

Lemma 1.4 *A weight w on an extended Dynkin diagram Γ is of the form $v \mapsto (u, v)$ for an element u in the root lattice precisely when w is zero on the Kodaira-Néron cycle and its restriction to the Dynkin diagram from which Γ is extended is of the form $v \mapsto (u, v)$.*

PROOF: This follows from the fact the vertex added to the Dynkin diagram to make Γ appears with multiplicity 1 in the Kodaira-Néron cycle F so that every element on the root lattice may be uniquely written as a sum of a multiple of F and an element supported on the Dynkin diagram. \square

We shall need to classify the admissible weights and the following result is the main tool in doing that. The precise classification will be left to the two following sections (even though we shall give no formal result describing the classification but only the consequences for fibres of genus 1 fibrations).

Proposition 1.5 *Let Γ be a simply-laced extended Dynkin diagram and w an admissible weight with excess e and fibre weight m .*

- i) *If Γ equals \tilde{A}_n or \tilde{E}_6 then either w has constant value 0 or w takes only values 1 and -1 and adjacent vertices have different weights. In both cases the fibre weight is 0 and the excess is 0.*
- ii) *If Γ equals \tilde{D}_4 or \tilde{E}_n , $n = 7, 8$, then $e + m = 2f$ where f is minus the weight of the vertex of degree > 2 .*
- iii) *If Γ equals \tilde{D}_n , $n > 4$ then $e + m = f_1 + f_2$, where f_1 and f_2 are minus the weights of the two vertices of degree 3.*
- iv) *We always have that $f, f_1, f_2 \leq 1$ except when $\Gamma = \tilde{D}_4$ in which case $f = 2$ is also possible for the weight giving the central vertex weight -2 and the others weight 1 and which has fibre weight 0.*

v) If w' is the complement of a representing element for w and v is a vertex that appears with multiplicity 1 in the Kodaira-Néron cycle, then $w'(v) = 0$.

vi) If the excess is zero, then either w is zero (and then so of course is the fibre weight) or w takes only values 1 and -1 and adjacent vertices have different values. In that case the fibre weight is 0 or 2.

- If w is non-zero and of excess and fibre weight 0, then Γ is \tilde{D}_{4n+1} or \tilde{E}_6 . The weight is uniquely determined up to automorphisms of Γ in the \tilde{D}_{4n+1} case and there are two of them in the \tilde{E}_6 case, differing by multiplication by -1 .
- If w is of excess 0, and fibre weight 2, then Γ is \tilde{D}_{2n} , \tilde{E}_7 , or \tilde{E}_8 and w is uniquely determined by Γ . If u is a representing element and w' its complement, then w' has support in the following vertex: In the \tilde{D}_{2n} case the vertex is the central vertex, i.e., the vertex fixed under all automorphisms and in the \tilde{E}_7 case the vertex must be the degree 1-vertex adjacent to the degree 3 vertex. Conversely, for those vertices there is a unique representing element whose complement has support in it.

PROOF: Using the notations of Definition-lemma 1.2 we have $nF = E - V$. Applying w and rearranging we get $e + nm = -w(V)$. By (1.2) we have that $V = 0$ when Γ equals \tilde{A}_n or \tilde{E}_6 and as the admissibility implies that $e, m \geq 0$ we get $e = m = 0$. This implies that the excess of every edge is 0 (as the support of the excess divisor equals the set of all edges) and hence an edge is either an s -edge or an n -edge. As the graph is connected this implies that all edges are of the same type.

As for ii) and iii) they say that $-w(V)$ equals $2f$ and $f_1 + f_2$ respectively which follows from (1.2).

Considering iv) we can write E as $E' + E''$, where E' is the sum (with multiplicities) of the edges containing one of the vertices of degree > 2 . We assume now that there is only one such vertex, v , the other case being similar. We thus get that $e = e' + e''$, where $e' = -w(E')$ and $e'' = -w(E'')$ both of which are non-negative. As all weights of vertices are ≤ 1 we get that the excess of an edge on which v lies is $\geq f - 1$ and hence we get that $e' \geq t(f - 1)$, where t is the sum of all the multiplicities in the excess divisor of the edges on which v lie. This t is 4, 6, and 8 for Γ equal to \tilde{D}_4 , \tilde{E}_7 , and \tilde{E}_8 respectively. We thus get $2f \geq t(f - 1) \geq 4(f - 1)$ and if $t \geq 5$ this gives $2f \geq 5(f - 1)$ and hence $f \leq 5$. When $t = 4$ we get the extra possibility that $f = 2$ in which case we have equality everywhere so that the fibre weight is 0 and all vertices but the central one have weight 1 which together with $f = 2$ gives the weights.

Let v be a vertex v appearing with multiplicity 1 in f and let w' be the complement of a representing element. By assumption v does not appear in u . Then either a vertex appearing in u is adjacent to v in which case $w(v) \geq w'(v) + 1$ or there is a vertex v' adjacent to v which is not in the support of u and then $w(v + v') \geq w'(v) + (u, v') \geq w'(v)$ both of which implies that $w'(v) = 0$. This proves v).

To prove vi) we note that a vertex can not lie on both an s - and an n -edge as the weight would simultaneously have to be zero and non-zero. Hence if all edges are non-excessive they are all of the same type. If they are all s -edges then $w = 0$ and if they are all n -edges their values alternate and so they are all determined by the value on a single vertex which gives only two possibilities differing by a sign. It is easily checked that for such weights the fibre weight is 0 or ± 2 and 0 precisely when Γ is \tilde{D}_{2n+1} or \tilde{E}_6 . The existence in the \tilde{E}_6 -case will be done in Theorem 3.1 so only existence in the \tilde{D}_{4n+1} and non-existence in the \tilde{D}_{4n+3} case remains. We start by giving a representing element with rational coefficients. For that define the weight t by $t(v) = \lfloor d/2 \rfloor$, where d is the distance from v to the degree 1-vertices that are either the attached vertex or the one lying “on the same side” as the attached vertex. Let w be the weight defined by $w(v)$ equal to $t(v)$ if v is not a degree 1 vertex on the “opposite side” of the attached vertex and equal to $n/2$ and those two vertices. Hence if n is even a representing element exists. In the case when n is odd any other representing element is equal to the sum of the given one and a rational multiple of the Kodaira-Néron cycle. It is clear however that no such sum can be integral.

Assume now that w is non-zero and excess 0, and weight 2. When the fibre weight is non-zero, the sign of w is determined by the admissibility condition so that the uniqueness of the admissible weight is clear. As for representing elements consider such an element u and its complement w' . If f is the Kodaira-Néron cycle then we have $w'(f) = 2$. As $w'(v)$ is zero for v a vertex of multiplicity 1 in f and as the support of f is the whole graph we get that w' is supported on a single vertex of multiplicity 2 in f . Thus given the supporting vertex, w' is uniquely determined and hence so is u up to a multiple of f . Let us first consider necessary conditions. For the \tilde{D}_{2n} case we first define t by $t(v) := [(d+1)/2]$, where d is the distance to the boundary, i.e., the smallest distance to a vertex of degree 1 and $[-]$ the integer part. It is easily verified that if $u := \sum_v t(v)v$, the sum running over the vertices, then it is a representing element for w whose complement has support in the central vertex. From this it also follows that this weight fulfills all the conditions of admissibility. For any other vertex of multiplicity 2 in Kodaira-Néron cycle consider the difference of the weight that is its characteristic function and the similar weight wrt the central vertex. There is then a representing element for that vertex precisely when this difference is represented by the scalar product by an element in the root lattice which has support in the vertices appearing with multiplicity 2 in the Kodaira-Néron cycle (the last because of condition 6 for admissibility). It is always representable by the difference of the fundamental weights for the two vertices and a glance at the table of those weights (cf. [GrLie4-6, Planche IV]) reveals that this difference always has a non-zero coefficient for some degree 1-vertex (except of course when this difference is zero). For the \tilde{E}_7 case there are, up to isomorphism, two vertices of multiplicity two in the Kodaira-Néron cycle and one of them has a fundamental weight in the root lattice and the other one doesn't (cf. [GrLie4-6, Planche VI]) so that at most one of the choices has a representing element. In the \tilde{E}_8 case there are two such vertices but only one of them is seen to have a representing element with support in the vertices appearing with multiplicity > 1 in the Kodaira-Néron cycle. The existence for the \tilde{E}_7 and \tilde{E}_8 cases is given in Theorem 2.2. This finishes the proof of vi). \square

2 Quasi-elliptic fibres

We shall now consider quasi-elliptic fibrations $\pi: X \rightarrow S$ in characteristic 2 and their conductrices. Note that some of our non-existence results can be proven by other means (cf. [CD89]) but we wanted to give a unified proof. We start with two observations concerning the curve of cusps.

Lemma 2.1 *Let $f: X \rightarrow S$ be a quasi-elliptic fibration with S a smooth curve over an algebraically closed field \mathbf{k} of characteristic 2. Then if R is the curve of cusps of f , R is smooth, the restriction of f to R is purely inseparable of degree 2 and the reduced inverse image of R on this normalisation maps by degree 1 to R . Furthermore, the normalisation of the pullback of f by the Frobenius map of S is a genus 0 fibration.*

PROOF: If we can prove that the restriction of f to R is of degree 2 then we get that it is non-singular as it intersects a fibre in a singular point so that if R had a singularity its intersection number with the fibre going through the singularity would be larger than 2. Hence we may assume that S is the spectrum of a separably closed field K of transcendence degree 1 over \mathbf{k} . In that case we have a point on X and thus f has a Weierstrass form $\{y^2 = x^3 + ax + b\}$. The curve of cusps is then defined by $x^2 = a$ and $y^2 = b$. After pulling back by the Frobenius map a and b become squares; $a = c^2$ and $b = d^2$ which to begin with implies that the curve of cusps is inseparable of degree 2. Furthermore, we may write the Weierstrass equation as $(y+c)^2 = x(x+d)^2$ and hence the coordinate ring of the normalisation is $K^{(2)}[s]$ with $y = s^3 + cs + d$ and $x = s^2$ which makes the fibration of genus 0 and the inverse image of R is defined by s and is hence of degree 1 over R . \square

We can now give the theorem describing the possible conductrices for a quasi-elliptic fibre.

Theorem 2.2 *Let S be the spectrum of a discrete valuation ring of characteristic 2 and $\pi: X \rightarrow S$ be a minimal quasi-elliptic fibration and X regular. Then if the special fibre of π is reducible*

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Consider first the \tilde{A}_1^* case when by Proposition 1.3 the weight given by intersection with the conductrix is semi-admissible. Let u be a representing cycle and assume that it is non-zero. As the Kodaira-Néron cycle is the sum of the two components u is at most supported on one

on one of them. Let v be the other vertex. As the intersection of the two vertices is 2 we get $w(v) \geq (u, v) \geq 2$ contradicting admissibility. Thus $w = w'$ and hence w takes values 0 or 1 the two vertices. If it takes the value 1 on both, the multiplicity is 2 and if it takes the value 1 on only one of them the multiplicity is 1. The multiplicities are then determined by Lemma 0.8.

We are hence left with the case where the fibre has a dual graph. By Proposition 1.3 intersection with the conductrix gives a weighting of fibre weight 1 or 2 on the dual graph of the fibre. If the excess is zero, the weight and possible representing elements are described in the proposition. We want to determine in that case the self-intersections. However, by Lemma 0.8 the only ambiguity is when the weight is -1 in which case we could have self-intersection -3 or -4 . However, a curve of type -3 would have s -invariant 1 and hence, again by the lemma, could not meet a curve with weight 1 as it has s -invariant 1 and r -invariant 0. Now, every vertex of weight -1 meets a vertex of weight 1. Hence all weight 1 curves are of type -4 .

We may therefore assume that the excess is strictly positive. Proposition 1.5 then gives us that Γ equals \tilde{D}_n or \tilde{E}_n , $n = 7, 8$, and $e + m = f_1 + f_2$ in the \tilde{D}_n , $n > 5$ cases and $e + m = 2f$ in the \tilde{D}_4 , \tilde{E}_7 , and \tilde{E}_8 cases. In all cases $f_1, f_2, f \leq 1$ and as $e, m > 0$ we get $e = m = f = f_1 = f_2 = 1$. Hence there is exactly one excessive edge k and k has to appear with multiplicity 1 in the excess cycle and the excess for k is 1. Consider the graph Γ' obtained by removing that edge (but not the vertices on it). Then Γ' has only edges of excess zero and thus each connected component of it consists exclusively of either s - or n -edges and in particular all vertices of Γ have weight -1 , 0, or 1. As Γ is a tree Γ' has two components and both components can not have edges of the same type as vertices on an s -edge have even weight and vertices on an n -edge have odd weight and k has excess 1. Hence there is one component which consists of s -edges and another which consists of n -edges. Then one of the vertices on k has to have weight -1 and the other weight 0 as otherwise the excess would be -1 .

Consider first the \tilde{D}_{2n} case. As f or f_1 and f_2 are 1 we have that the degree 3-vertices belongs to the n -component and hence the s -component consists only of one degree 1-vertex which we can assume to be the attached vertex. On the other hand, if u is a representing cycle and w' its complement then as $w'(f) = 1$, f being the Kodaira-Néron cycle, we have that w' is supported on a degree 1-vertex. This can be any of the four degree 1-vertices. The difference of the complements of two such choices is then given by the difference of the two fundamental weights (or the dual of largest root in the case of the attached vertex). The only relations modulo the root lattice is that the weight corresponding to the attached vertex is zero, that the fundamental weights are all of order 2, and that the sum of the three fundamental weights is zero (as is easily seen from [GrLie4-6, Planche IV]). This implies that the difference of two of them can never be in the root lattice and hence at most of these cases have a representing element. Consider now the weight t given by $t(v) = [(d + 1)/2]$, where d is the distance to the attached vertex, when v has degree > 1 , $t(v) = 0$ for the degree 1-vertices on the “opposite side” of the attached vertex, $t(v) = [n/2]$ for the attached vertex, and $t(v) = [(n - 1)/2]$ for the fourth degree 1-vertex. It is then easily verified that this is a representing element.

In the \tilde{E}_7 and \tilde{E}_8 cases there is up to isomorphism a unique vertex of degree 1 in the Kodaira-Néron cycle and it is easily verified that the claimed element is representing.

Finally, the self-intersection types are as claimed as the only ambiguity is type -4 and -3 . However, a vertex of type -3 can, by Lemma 0.8 not be adjacent to one of type -1 or -2 . \square

Remark: It is a striking *a posteriori* fact that given the multiplicity and the type of the fibre there is at most one possibility for the conductrix. We have no *a priori* explanation for this.

We may use these results to show that the existence of certain quasi-elliptic fibrations forces exceptionality.

Corollary 2.3 *Let X be an Enriques surface in characteristic 2.*

- i) *X can not have a simple \tilde{E}_8 -fibre of a quasi-elliptic fibration in the support of its conductrix.*
- ii) *If X has a quasi-elliptic fibration with either a \tilde{E}_7 - or a \tilde{E}_8 -fibre (simple or double) then it is exceptional.*

PROOF: For the first part we see from the theorem that for such a fibre, the bi-conductrix B would contain the fibre contradicting that by Proposition 0.5 $h^0(B) = 1$.

As for the second part the case of double fibres follows directly from the theorem as two times the conductrix visibly contains a half fibre. In the case of a simple \tilde{E}_7 -fibre the theorem shows that the contribution to the conductrix supported on that fibre plus the curve of cusps form a \tilde{E}_6 -configuration which then is the half-fibre (“half” as it has intersection 1 with a curve) of a (necessarily) elliptic fibration. Twice the conductrix contains that half-fibre so the surface is exceptional by Proposition 0.5. In the case of a simple \tilde{E}_8 -fibre the theorem shows that we are in one of two cases but one of those has just been excluded. Hence the curve of cusps meets the vertex of degree 1 that has multiplicity 2 in the Kodaira-Néron cycle. Now, by Lemma 2.1 the curve of cusps together with the intersection of Γ with the support of the conductrix forms a $T_{2,4,5}$ -configuration and in particular contains a $T_{1,4,4} = \tilde{E}_7$ which is the half-fibre of a genus 1-fibration (again “half” as the remaining curve in $T_{2,4,5}$ -configuration) has intersection number 1 with the Kodaira-Néron cycle). Twice the conductrix then contains the half-fibre and so the surface is exceptional. \square

In the case of a double fibre the theorem determines the conductrix only up to a multiple of the half fibre. The following result relates that multiple to a more familiar invariant.

Proposition 2.4 *Let $f: X \rightarrow S$ be a quasi-elliptic fibration with S the spectrum of a discrete valuation ring in characteristic 2 and \overline{X} be the normalisation of the pullback of f by the Frobenius map. Then \overline{X} has only rational double points as singularities and the length of the torsion of $R^1 f_* \mathcal{O}_X$ equals the largest multiple of the special fibre that is contained in the conductrix.*

PROOF: Let $f': X' \rightarrow S$ be the pullback of f by the Frobenius map of S , $\tau: \overline{X} \rightarrow X'$ its normalisation, and $\rho: \tilde{X} \rightarrow \overline{X}$ a minimal resolution. Also, put $\tilde{f} := \tau \circ f'$ and $\hat{f} := \rho \circ \tilde{f}$. Then \hat{f} is the blowing up of a smooth genus 0 fibration so that $R\hat{f}_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_S$. On the other hand we have a distinguished triangle $\rightarrow \mathcal{O}_{\overline{X}} \rightarrow R\rho_* \mathcal{O}_{\tilde{X}} \rightarrow \mathcal{F}[-1] \rightarrow$, where we have put $\mathcal{F} := R^1 \rho_* \mathcal{O}_{\tilde{X}}$. Using that \mathcal{F} has finite support and applying $R\hat{f}_*$ we get a short exact sequence

$$0 \rightarrow R^1 \tilde{f}_* \mathcal{O}_{\overline{X}} \rightarrow R^1 \hat{f}_* \mathcal{O}_{\tilde{X}} \rightarrow \tilde{f}_* \mathcal{F} \rightarrow 0,$$

and as $R^1 \hat{f}_* \mathcal{O}_{\tilde{X}}$ we get $\mathcal{F} = 0$ and $R^1 \tilde{f}_* \mathcal{O}_{\overline{X}} = 0$. Consider now the inclusion $\mathcal{O}_{X'} \hookrightarrow \tau_* \mathcal{O}_{\overline{X}}$ with cokernel that we call \mathcal{G} . By the vanishing of $R^1 \tilde{f}_* \mathcal{O}_{\overline{X}}$ and that $\tilde{f}_* \mathcal{O}_{\overline{X}} = \mathcal{O}_S = f'_* \mathcal{O}_{X'}$, we get that $f'_* \mathcal{G} = R^1 f'_* \mathcal{O}_{X'}$. Now, on the one hand, by flat base change we have $R^1 f'_* \mathcal{O}_{X'} = F^* R^1 f_* \mathcal{O}_X$ and hence $F_* R^1 f'_* \mathcal{O}_{X'} = \mathcal{O}_S \otimes_{\mathcal{O}_S} R^1 f_* \mathcal{O}_X$, with \mathcal{O}_S being a module over itself through the Frobenius map F . On the other hand we have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & F_* \mathcal{O}_{X'} & \longrightarrow & \mathcal{L} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & F_* \tau_* \mathcal{O}_{\overline{X}} & \longrightarrow & \mathcal{L}(A) \longrightarrow 0, \end{array}$$

where \mathcal{L} being the pullback of a line bundle on S is trivial and more precisely equal to $\pi \mathcal{O}_S$ where π is a uniformiser for S . This gives a short exact sequence

$$0 \rightarrow \pi \mathcal{O}_X \rightarrow \pi \mathcal{O}_X(A) \rightarrow F_* \mathcal{G} \rightarrow 0$$

and hence a long exact sequence

$$0 \rightarrow \pi \otimes f_* \mathcal{O}_X \rightarrow \pi \otimes f_* \mathcal{O}_X(A) \rightarrow (\mathcal{O}_S \oplus \pi \mathcal{O}_S) \otimes_{\mathcal{O}_S} R^1 f_* \mathcal{O}_X \rightarrow R^1 f_* \mathcal{O}_X$$

and it is easily seen that the last map is just the projection on the first factor giving a short exact sequence

$$0 \rightarrow f_* \mathcal{O}_X = \mathcal{O}_S \rightarrow f_* \mathcal{O}_X(A) \rightarrow R^1 f_* \mathcal{O}_X \rightarrow 0.$$

Now, we can write $A = F + R$, where F has support in the special fibre and R is the curve of cusps. It is clear that for the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(F)$ given by the section F we have


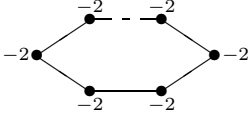
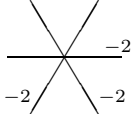
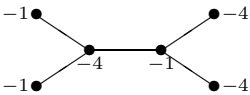
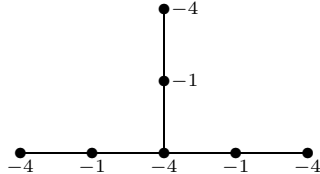

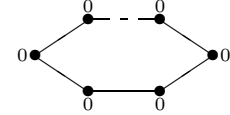
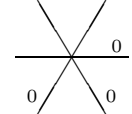
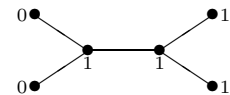
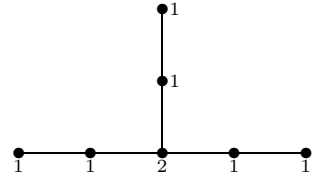
$f_*\mathcal{O}_X = \mathcal{O}_S \hookrightarrow f_*\mathcal{O}_X(F) = \pi^{-n}\mathcal{O}_X$, where n is the largest multiple of the special fibre that is contained in the conductrix. Furthermore, $\mathcal{O}_X(A)/\mathcal{O}_X(F)$ is flat over S and hence the S -torsion of $R^1f_*\mathcal{O}_X$ equals $f_*\mathcal{O}_X(F)/f_*\mathcal{O}_X \cong \mathcal{O}_S/\pi^n\mathcal{O}_S$. \square

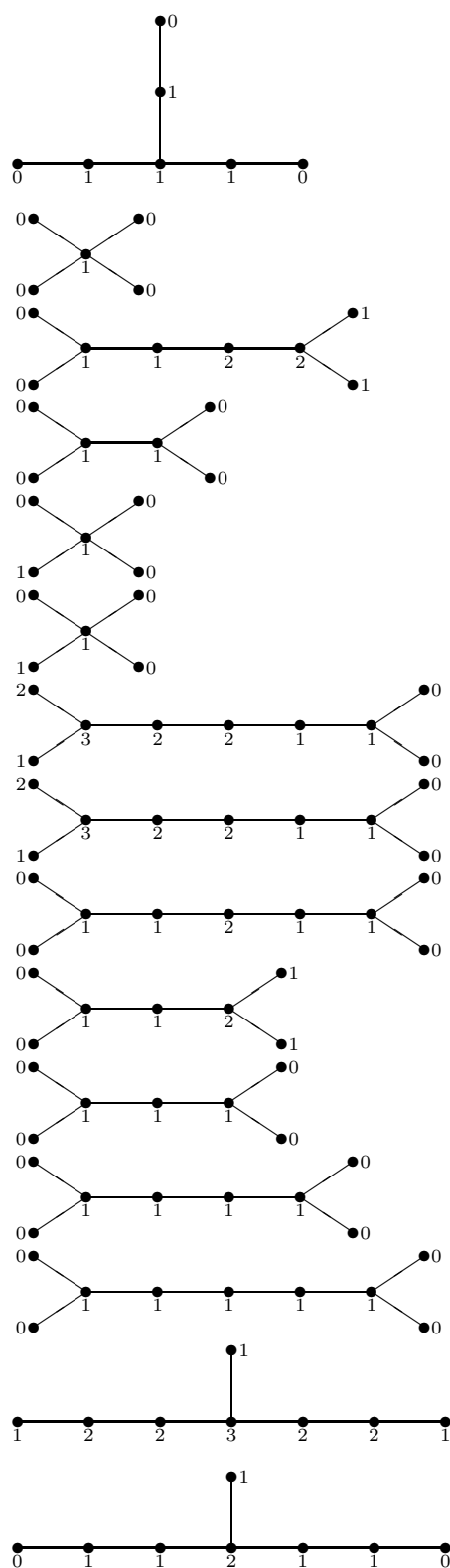
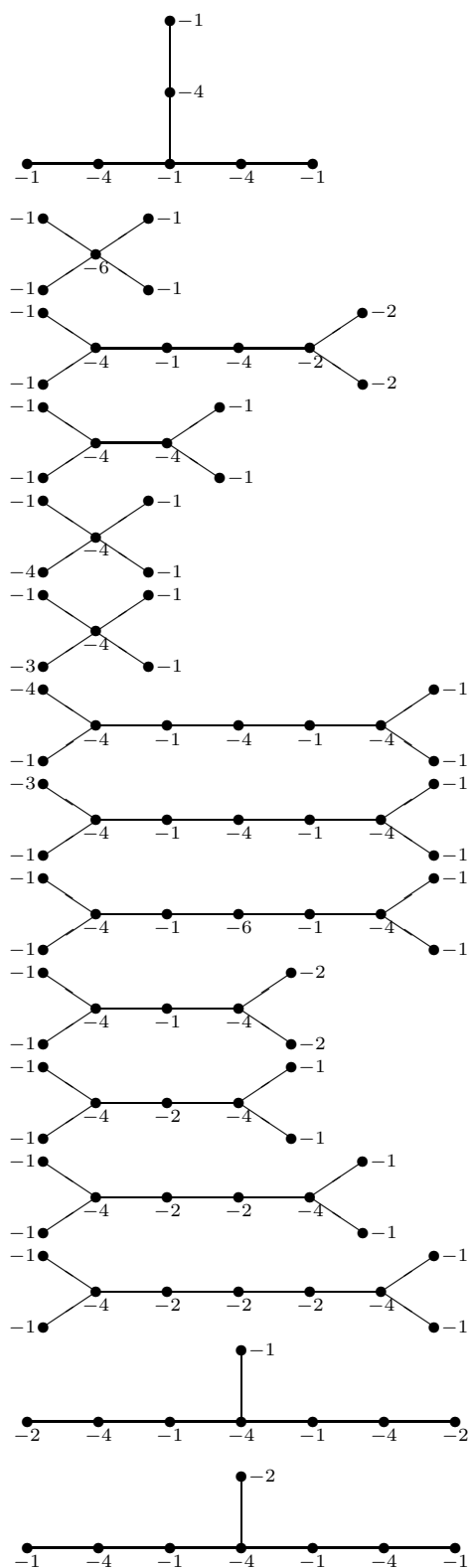
Remark: Note that the conductrix of a genus 1 fibration with a double fibre always contains a half-fibre as the equation for a fibre is divisible by the square of the half-fibre and one can construct a partial normalisation of the pullback by the Frobenius map by dividing by the equation of the half-fibre. The fibre is then wild precisely when the conductrix for this partial normalisation contains a half-fibre.

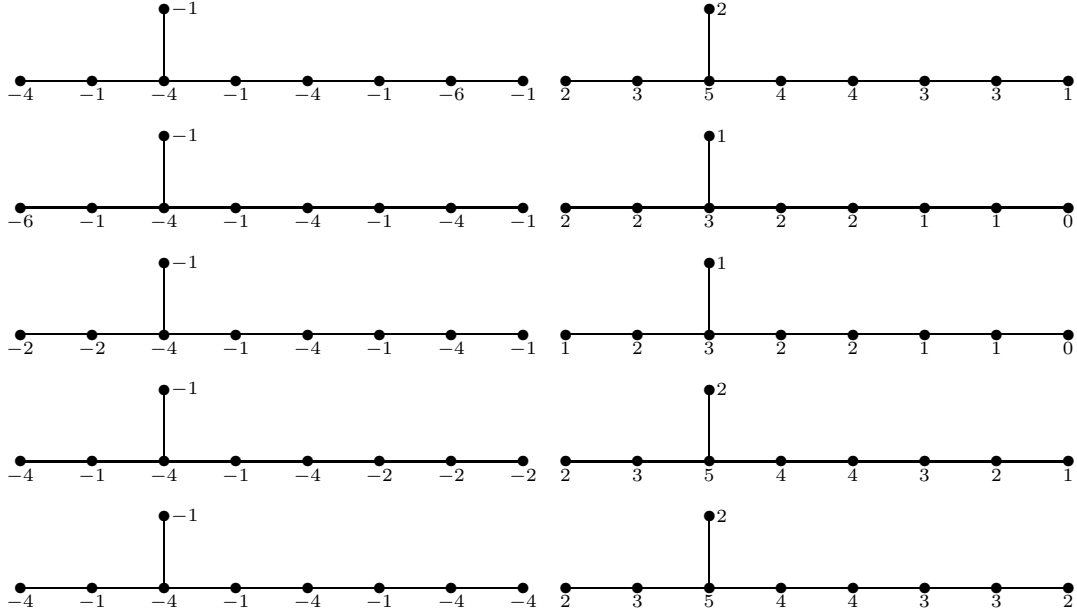
3 Elliptic fibres

Let us first take care of the \tilde{A}_n and \tilde{A}_2^* cases. Then the dual graph is \mathbf{A}_n and we know from Proposition 1.5:i that if the weight is not everywhere zero it takes the values 1 and -1 on alternating vertices and the multiplicity is zero. Let u be a representing element. Then $(u, v) = \pm 1$ for every vertex. Furthermore, as the Kodaira-Néron cycle contains every vertex root exactly once the support of u is not the whole graph. Let v be a vertex not in the support. Then $(u, v) \geq 0$ and hence it must equal 1. This means that it is only adjacent to one vertex in the support. Let v' be the vertex adjacent to v not in the support. By the same argument we have $(u, v') = 1$ which contradicts that the signs of $(-, u)$ alternates.

Theorem 3.1 *Let S be the spectrum of a discrete valuation ring of characteristic 2 and $\pi: X \rightarrow S$ be a minimal elliptic fibration and X regular. Then if the special fibre of π is reducible we have the following possibilities, where A_s denotes the special fibre and the conductrix up to a multiple of the Kodaira-Néron cycle of the special fibre, F gives the self-intersection type (see 0.8) of the curves of the special fibre. Also only the fibres with at most 9 components have been displayed, the others can be inferred from statements in the proof or extrapolated from the display.*

| F | A_s |
|---|--|
|      |      |





PROOF: We may assume that the fibre is reducible as otherwise the theorem is trivial.

Consider first the case of a \tilde{A}_1^* fibre. A reduced representing element for the weight is a non-negative multiple of one of the vertices but if it is a strictly positive multiple it has intersection > 1 with the other vertex and hence does not represent a semi-admissible weighting as it should by Proposition 1.3.

By Proposition 1.3, when the fibre has a dual graph, intersection with the conductrix gives rise to an admissible weighting.

Consider now the case when the conductrix is zero and the fibre is reducible. Then the self-intersection type of all components is -2 . When the fibre has a dual graph there can be no vertices of degree 3 as then, by Lemma 0.8, its r -invariant would have to be > 2 . In the case the fibre is of type \tilde{A}_n all intersections between two components have to lie below a singularity and that singularity has to be of type A_1 as otherwise we would get a dual graph that is not a Dynkin diagram. As the intersection points already account for the full r -number we see that we get a \tilde{A}_{2n} fibre. Similarly, a fibre of type \tilde{A}_2^* would have a A_1 -singularity on the intersection of the three components and then the only possibility is that there is one more A_1 singularity on each component, giving a \tilde{E}_6 fibre.

When the fibre is of type \tilde{A}_n or \tilde{A}_2^* we get from Proposition 1.5:i that the excess is zero and that implies that the weight is zero. We may therefore assume that the fibre is of type \tilde{D}_n or \tilde{E}_n .

We now assume that the weight is not zero and divide up our analysis first according to the excess of the weighting.

Excess 0:

The case when the excess is zero and the weighting is non-zero is handled by Proposition 1.5:i. The self-intersections are determined — as usual with the aid of Lemma 0.8 — as every vertex with self-intersection type -3 or -4 is adjacent to one of self-intersection type -1 and hence must be of type -4 .

We can thus assume that the excess is strictly positive. With the notations of the proposition we then have $e = 2f$ or $e = f_1 + f_2$. By the proposition there is one single possibility in the \tilde{D}_4 case for which $f = 2$ whose self-intersections are determined by (0.8). We can thus assume that $f, f_1, f_2 \leq 1$.

Excess 1:

As 1 is not even, the fibre must be of type \tilde{D}_n , $n > 4$ and we may assume $f_1 = 1$ and $f_2 = 0$. As the excess is 1 there is exactly one excessive edge and removing that gives a subgraph whose

components consists solely of edges of the same type, s -edges or n -edges. Now, as one of the vertices of degree 3 has weight 0 and the other weight -1 we get that they belong to different components and hence the excessive edge lies between them. Furthermore, the degree 3 vertex in the s -component can not be interior in that component as otherwise it would be adjacent to more than two vertices of weight 0 contradicting admissibility. Hence, the edge connecting that degree 3 vertex with a degree 2 vertex has to be the excessive one. This determines the weight, as the other vertex on the excessive edge has to have weight -1 and then the weights have to alternate between 1 and -1 . This excludes the case when n is even as in that case the other degree 3 vertex can not have weight -1 as it must as $f_1 = 1$. We may thus assume that n is odd. We now define a (rational-valued) weighting as follows: Let t be the weight given by $t(v) = [(d+1)/2]$, where d is the distance to the degree 1 vertices on the side of the attached vertices and let w be the weight that takes the value $t(v)$ on all vertices but the degree 1 vertices on the opposite side of the attached vertex and value $(n-3)/4$ on these vertices. A moment's thought shows that w gives the weighting we are looking for and it is easily seen that adding a (rational) multiple of the Kodaira-Néron cycle can give an element of the root lattice precisely when $n \equiv 3 \pmod{4}$.

Excess 2:

This forces $f = f_1 = f_2 = 1$. Furthermore, the type of the fibre is either \tilde{D}_n , \tilde{E}_7 , or \tilde{E}_8 . We now consider those different types.

- \tilde{D}_n : Assume first that there is one edge of excess 2. Removing that edge gives again a graph whose components have edges of the same type, s - or n -edges. In particular, the weight of a vertex is -1 , 0 , or 1 and hence the vertices on the excessive edge both have weight -1 . If the excessive edge lies between the two degree 3 vertices, then as both degree 3 vertices have weight -1 and apart from the excessive edge the weights must alternate, we get that n must be odd. An argument similar to the previous ones shows that the representing element is in the root lattice precisely when $n \equiv 1 \pmod{4}$. Just as before there can be no vertex of self-intersection type -3 as every vertex of weight -1 is adjacent to one of weight 1 . If the excessive edge instead is one of the degree 1 vertices we get first that n must be even as the two degree 3 vertices have weight -1 and the weights between them must alternate. One then shows that in order for the representing element to be in the root lattice we must further have that n is divisible by 4.

The next case is still of type \tilde{D}_n but with two edges of excess 1. Removing them gives three components all of which have edges of excess zero. Assume first that there is a vertex v of weight < -1 . Then it has to be isolated in its component, of weight -2 and adjacent only to vertices of weight 1 . As the two degree 3 vertices have weight -1 this means that v has to lie between them. We also have that the other weights are -1 or 1 and that they alternate. This forces n even and $n \geq 8$. Apart from it is easily seen that there are no further restrictions. We see also that the self-intersection type of v must be -6 as the s -invariant must be 2 by Lemma 0.8.

Hence, we may assume that all vertices have weights -1 , 0 , or 1 (and still a graph of type \tilde{D}_n). Again we get three components after having removed the two excessive edges. If the two degree 3 vertices lie in the same component then two of the degree 1 vertices have weight 0 and the rest of the graph comprise one component. Furthermore, n is even as otherwise the two degree 3 vertices can not have weight -1 . Up to isomorphism this gives two possibilities: Either the two weight 0 vertices lie on the “same side” or they don't. In the first case define the rational valued weight t by $t(v) = [(d-1)/2]$, where d is the distance to the attached vertex, when v is not a degree 1 vertex, $t(v) = 0$ if v lies on the same side as the attached vertex, and $t(v) = (n-2)/4$ if not. This weight has the right properties and is integer-valued precisely when $n \equiv 2 \pmod{4}$. As for the second case we see that the multiplicity of a degree 3 vertex in a representing cycle would have to be both odd and even.

In the case when the two degree 3-vertices lie in different components we then have a “line”

of weight 0 vertices among the degree 2 vertices. In the case when all degree 2 vertices have weight 0 a representing element is having multiplicity 1 at all vertices of degree > 1 and zero at those of degree 1. In general it is easily seen (using for instance [GrLie4-6, Planche IV]) that a representing element exists exactly when the number of vertices of weight 0 is congruent modulo 4 to the number of vertices of degree 2 (i.e., $n - 5$).

Hence either there are two excessive edges both occurring with multiplicity 1 in the excess cycle or one excessive edge either occurring with multiplicity two in the excess cycle or being itself of excess 2 and occurring with multiplicity 1 in the excess cycle.

- \tilde{E}_7 : As the total excess is 2 there are at most two excessive edges. Note that generally the weights on the component of the graph of non-excessive edges that contain the degree 3 vertex are determined as the weight of the degree 3 vertex is -1 . If an edge that is of distance 1 from the boundary (where an edge with a boundary vertex on it is of distance zero to the boundary) is excessive then its inner vertex has weight 1 and hence its outer vertex has weight -2 forcing the other edge on which it lies also to be excessive.

Assume that there are exactly two excessive edges. Then their excess must be 1 and they must both appear with multiplicity 1 in the excess cycle. Hence if an edge of distance 1 to the boundary is excessive, the weights of the whole graph are determined and are seen to give a weighting that can not be represented by an element of the root lattice (though by an element in the rational root lattice). Hence we may assume that only the outer edges of the long arms are excessive and this determines a weight which is seen to be represented by an element of the root lattice.

Assume now that there is only one excessive edge. This is either an edge of excess 1 and with multiplicity 2 in the excess cycle or with excess 2 and multiplicity 1 in the excess cycle. In either case there is exactly one weight. In the first case the cases are detected by which of the three neighbours to the degree 3 vertex are of weight 0. For the degree 1 neighbour there is a representing element, for the others the weight is not even zero on the Kodaira-Néron cycle. For the case of an excess 2 edge it can not be of distance 1 to the boundary as that would give another excessive edge and hence it must be a boundary edge on one of the long arms which determines the weight, which is seen not to be represented by an element of the root lattice.

- \tilde{E}_8 : We proceed as in the \tilde{E}_7 case.

Assume that there are exactly two excessive edges each thus having excess 1. As there are exactly two edges appearing with multiplicity 1 in the excess cycle this means that those two edges are the excessive edges and the weight is determined and it is indeed given by a representing element.

Assume that there is exactly one excessive edge. The weight is completely determined by which edge is excessive and that edge has multiplicity 1 or 2 in the excess cycle. The two edges on the length 2 arm both have representing cycles. If the second edge on the long arm (counted from the degree 3 vertex) were excessive, the multiplicity of its outer vertex would have to be -2 and hence also the next edge would be excessive. The third edge however gives rise to a representing cycle. If the fourth edge on the long arm were excessive, the multiplicity of its outer vertex would have to be -3 and hence also the next edge would be excessive. Finally, the outer edge on the long arm gives rise to an admissible weight.

We have thus taken into account all the possibilities and the proof is therefore finished. \square

Recall that a genus 1 fibration on an Enriques surface is said to be special if it has a genus 0 2-section. The curve of cusps of a quasi-elliptic fibration gives such a section so that quasi-elliptic fibrations are always special. We shall now use the theorem to investigate elliptic special fibrations.

Corollary 3.2 *Let X be an Enriques surface.*

i) *A double fibre of a special elliptic fibration on X is either irreducible, of types \tilde{A}_1^* , \tilde{A}_2^* , \tilde{D}_5 with the weighting associated to the conductrix of excess 0 or \tilde{E}_6 with the weighting having the whole fibre as its support.*

ii) *Conversely, if an elliptic fibration on X has a double \tilde{D}_5 or \tilde{E}_6 fibre for which the weighting associated to the conductrix has excess 0 resp. with the whole fibre as its support, then it is special.*

PROOF: Assume first that R is a special 2-section of an elliptic fibration and F is a fibre which may be assumed to be reducible not of type \tilde{A}_1^* or \tilde{A}_2^* . In particular F has a dual graph. As F is a double fibre we have that $(C, R) = 1$, where C is the Kodaira-Néron cycle of F . In particular, R meets only one component of F and it meets that component in one point and has a transversal intersection in it. Hence the configuration of -2 -curves given by the union of the curves of F and R has a dual graph Γ' and that graph is obtained (up to isomorphism) from the dual graph Γ of F by attaching the vertex associated to R at the attached vertex of the extended Dynkin diagram Γ . The conductrix gives rise to a weighting w on Γ' whose restriction to Γ is given by one of the possibilities of the theorem and whose value on R is ≥ 0 as R is not in the support of the conductrix and is ≤ 1 by Lemma 0.8. Furthermore, the value on the edge root on the edge on which R lies is, again by (0.8), ≤ 0 . If the restriction of w to Γ is non-zero then as the conductrix is connected (cf Proposition 0.5), supported in the fibres of the fibration and contains components of F , it is completely contained in F and is given by the theorem, taking into account that by Proposition 0.5 the part of the conductrix lying in a fibre is reduced.

Consider first the case when the restriction of w to Γ is zero. Then by the theorem $\Gamma = \tilde{A}_n$ for some n . By Lemma 0.8 we must have $w(R) = 0$ and the intersection point of R and F must be blown up under the minimal dissolution of X . As F must have type \tilde{A}_n this is not possible as, again by (0.8), all the intersection points of components of F are blown up during the minimal dissolution and exactly 2 points of each component are blown up. This gives a contradiction as the single component that R meets would have three points blown up.

Hence we may assume that the restriction of w to Γ is non-zero and then $w(R)$ is given by the multiplicity of the attached vertex of some A of the theorem associated to the type of the fibre F . Hence that multiplicity must be either 0 or 1 and when it is 1 the self-intersection type of the attached vertex must be -4 or -6 and when it is 2 the self-intersection type must be -6 , -4 , -3 or -2 (as is required by Lemma 0.8). A look at the table of the theorem gives the desired conclusion.

Suppose conversely that X has an elliptic fibration with a double \tilde{D}_5 fibre and that the conductrix is as specified. Note that as the conductrix is supported in F its support forms a D_4 . Assume first that X is \tilde{E}_8 -special in the sense of [CD89, 3:§4]. In our terminology that is the same thing as saying that X has a special genus 1 fibration with a double fibre of type \tilde{E}_8 . By what has just been proven such a fibration is quasi-elliptic but by Theorem 2.2 such a fibration can not have a conductrix whose support forms a D_4 -configuration. By [CD89, Thm 3.4.1] there therefore is a second genus 1 fibration on X such that a half-fibre of it has intersection one with the half-fibre F . We may use this second fibration to compute the conductrix. Looking through the tables of Theorems 2.2 and 3.1 one concludes that it either is elliptic with a fibre of type \tilde{D}_5 with weight of excess 0, elliptic with a fibre of type \tilde{E}_6 with conductrix whose support is a D_4 in the fibre, or quasi-elliptic with a simple \tilde{D}_6 fibre. In the two last cases an appropriate curve of the new fibre gives a special 2-section of the original fibration and we may hence assume that we are in the first case. Furthermore, using Lemma 0.9, we conclude that the \tilde{D}_5 fibre of the new fibration must be a double fibre as the conductrix contains a curve that appears with multiplicity 1 in the Kodaira-Néron cycle. Hence we have two different \tilde{D}_5 -configurations whose intersection contains a common D_4 -configuration. This means that either they have one more curve in common, making the intersection a D_5 -configuration, or the intersection is the support of the conductrix. Hence, one of them has one or two curves outside of the other and they meet the first fibre in a degree 2 vertex. The intersection number of the two Kodaira-Néron cycles is

thus equal to the sum of the intersection numbers of these one or two curves of the second fibre with the Kodaira-Néron cycle of the first and as they meet the first in a curve appearing with multiplicity 2 in the Kodaira-Néron cycle this intersection number is at least 2 but by assumption this intersection number is supposed to be 1.

Suppose now instead that X has an elliptic fibration with a double \tilde{E}_6 fibre with the support of the conductrix being all of that fibre. Again by [CD89, Thm 3.4.1] there must be some other genus 1 fibration on X . This fibration can not be elliptic as then the support of the conductrix would be contained in a fibre but the support is already a fibre for the first fibration. Looking at the table for Theorem 2.2 we see that the only possibility for a quasi-elliptic fibration giving a conductor of the desired form is either one simple \tilde{D}_8 fibre and one simple \tilde{D}_4 fibre (both with weights of excess 0) or one simple \tilde{E}_7 fibre with weight of excess 0. Both provide a special 2-section through one of the components of a fibre not in the conductrix (the first one also giving too many components of fibres to be possible on an Enriques surface). \square

The theorem and the corollary allows us to give a first step towards characterising exceptional surfaces with elliptic fibrations.

Corollary 3.3 *Let X be an Enriques surface in characteristic 2. If a half fibre of an elliptic fibration is contained in the bi-conductrix then that fibre is a \tilde{E}_6 -fibre.*

PROOF: As the conductrix is connected (cf. Proposition 0.5) the support of the conductrix equals the double fibre in question. By the theorem we get that the fibre is of type \tilde{E}_n , $n = 6, 7, 8$ and what remains is to exclude the possibility of a \tilde{E}_7 or \tilde{E}_8 fibre. Consider first the \tilde{E}_8 case. By Corollary 3.2 the fibration is not special and hence by [CD89, Thm 3.4.1] there is another genus 1 fibration. It can not be elliptic because if it were the conductrix would be contained in a single fibre and as one extended Dynkin diagram can not be properly contained in another it would not be different from the given one. Thus it is quasi-elliptic and checking the list in Theorem 2.2 one sees that this is not possible (the quickest way to do this is probably to note that the degree 3 vertex has multiplicity 5 and in the quasi-elliptic case this only happens for a \tilde{E}_8 fibre, simple or double, but in neither of those cases is the support of the conductrix as apart from the part in the fibre one has at least also the curve of cusps). The case of an \tilde{E}_7 fibre is similar. \square

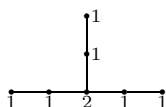
4 Genus 1 pencils on exceptional Enriques surfaces

We are now ready to give a description of exceptional Enriques surfaces in terms of which genus 1-fibrations they admit. In the case of surfaces of type $T_{2,3,7}$ and $T_{2,4,5}$ this is quite simple. The situation in the $T_{3,3,3}$ case is more complicated and the answer depends on the MW-rank of the surface. This forces us to divide the study in two pieces. First a lattice theoretic study which shows how to obtain every \tilde{E}_7 -configuration (genus 1 fibrations with such a fibre being the only ones that are troublesome) of elements in Num of the surface from a fixed configuration by a sequence of (very special) reflections. Then a determination of which of these sequences give an actual \tilde{E}_7 -configuration of curves is made. The last answer will then depend on the MW-rank.

In actuality it turns out that it is not the \tilde{E}_7 -configurations that are the ones to study but rather the $T_{4,4,4}$ -configurations and the following result gives a first explanation as to why this is.

Definition-Lemma 4.1 *Let X be an exceptional Enriques surface with an elliptic fibration on it.*

i) That fibration is special, has a double fibre of type \tilde{E}_6 , and the conductrix is supported on it and has the following form:



We shall call this \tilde{E}_6 -configuration the canonical $(T_{3,3,3})$ -configuration, \mathcal{C} . Irreducible components of reducible fibres of the elliptic fibration that are not part of the canonical configuration will be called extraneous components.

ii) Any other genus 1 fibration is quasi-elliptic with a simple fibre of type \tilde{E}_7 . One of the end vertices of the canonical configuration is the curve of cusps of it and the other vertices are contained in the given fibre.

iii) There is more than one genus 1-fibration on X .

iv) Any \tilde{E}_7 -configuration on X extends to a $T_{4,4,4}$ -configuration. In particular X has a $T_{4,4,4}$ -configuration. For each such surface we pick a $T_{4,4,4}$ -configuration, \mathcal{T} , and call it the standard $(T_{4,4,4})$ -configuration.

PROOF: The support of the conductrix is contained in one fibre as the fibration is elliptic and the conductrix is connected (cf Proposition 0.5). As one extended Dynkin diagram can not be properly contained in another that means that the support of conductrix, which by assumption will contain a configuration of the type of an extended Dynkin diagram, is equal to a fibre. Hence by corollary 3.3 the half fibre is of type \tilde{E}_6 and the conductrix is by Theorem 3.1 supported in that fibre and is as is claimed. Finally, the fibration is special by Corollary 3.2.

Suppose now that $\rho: X \rightarrow \mathbf{P}^1$ is another genus 1 fibration on X . If ρ is elliptic the conductrix has support in one fibre by (3.1) but that support, as we have just seen, is an \tilde{E}_6 -configuration. This forces the support of \tilde{E}_6 -configuration to be the whole fibre and hence ρ is the same fibration as π contrary to assumption. Hence, ρ is quasi-elliptic. The conductrix is then supported on the union of a finite number of fibres of ρ and the curve of cusps of ρ . The curve of cusps occurs with multiplicity 1 in the conductor whereas the central vertex of the \tilde{E}_6 -configuration occurs with multiplicity 2 in it. Hence the curve of cusps is not equal to that central vertex and therefore the central vertex lies in a fibre of ρ . Inspection of the tables of Theorem 2.2 shows that there are only two possible fibres of a quasi-elliptic fibration for which a vertex of degree 3 occurs with multiplicity 2 in the conductor; a double \tilde{D}_6 -fibre and a simple \tilde{E}_7 -fibre. The \tilde{D}_6 -fibre is not possible as then one of the arms of the dual graph of the conductrix would only consist of one non-central vertex whereas all the arms of \tilde{E}_6 consist of two such vertices. Hence any other genus 1 fibration is quasi-elliptic with a simple \tilde{E}_7 -fibre and the \tilde{E}_6 -configuration is the union of the curve of cusps plus the E_6 in the \tilde{E}_7 .

Now, it follows from [CD89, Thm 3.4.1] that an Enriques surface with only one genus 1 fibration has an \tilde{E}_8 -fibre in that unique fibration. This is not possible for π as it has an \tilde{E}_6 -fibre and the sum of the number of components of irreducible fibres minus the number of such components is at most 8 on an Enriques surface.

We thus know that there must be another genus 1 fibration and by ii) it has a simple \tilde{E}_7 fibre and it and the curve of cusps make up a $T_{4,4,3}$ -configuration. However, by what we have just shown it is quasi-elliptic and hence it must have another reducible fibre (as the number of components minus 1 of the fibres sum up to 8) and hence there is a -2 -curve in a fibre that meets the curve of cusps. By Lemma 0.8 it must meet it transversally and as it is in a fibre it meets no other curve of the $T_{4,4,3}$ -configuration. Adding it thus gives a $T_{4,4,4}$ -configuration. \square

Proof of Theorem A: That a surface with a quasi-elliptic fibration with an \tilde{E}_7 - or \tilde{E}_8 -fibre is exceptional follows from corollary 2.3 and that a surface with a special fibration a double \tilde{E}_6 -fibre is exceptional follows from Theorem 3.1 and Corollary 3.2. Conversely, assume that X is exceptional. By (0.5:v) the bi-conductrix contains a half fibre of a genus 1 fibration $\pi: X \rightarrow \mathbf{P}^1$.

Assume first that π is elliptic. By lemma 4.1 π is special with a double \tilde{E}_6 fibre.

Therefore we may assume that π is quasi-elliptic. Again consulting Theorem 2.2 we see that the only double fibres for which the support of the conductrix contains the fibre is a \tilde{E}_7 - or \tilde{E}_8 -fibre. This concludes the proof of the first part.

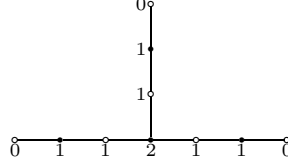
The support of the conductrix can be read off from (2.2) and (3.1).

Finally, the last part follows from corollary 2.3 and Theorem 3.1. \square

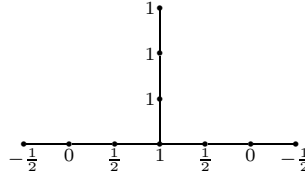
Proof of Theorem B: The characterisation follows from Proposition 0.5. \square

We shall go on to make a more precise study of the possible genus 1 fibrations on exceptional Enriques surfaces. The most difficult case is that of a surface of type $T_{3,3,3}$ and we need some preliminaries to handle that case.

We start by letting $Q_{4,4,4}$ be the lattice associated to the graph $T_{4,4,4}$ (i.e., with basis the vertices of it with scalar squares -2 and scalar product 1 or 0 depending on whether the vertices are connected or not). The rest of our definitions will use the following diagram.



We let $C_{4,4,4}$ be the element of M given by the vertex multiplicities specified by the diagram and let $Q'_{4,4,4}$ be the super-lattice of $Q_{4,4,4}(2)$ (the lattice obtained from $Q_{4,4,4}$ by scaling the scalar product by 2) spanned by $Q_{4,4,4}(2)$ and $1/2$ times every hollow vertex. Thus $Q'_{4,4,4}$ has a basis in bijection with the vertices of $T_{4,4,4}$, with scalar squares equal to -4 for filled vertices and -1 for hollow ones, and scalar product of distinct vertices the same as for $Q_{4,4,4}$. Let us further define the super-lattice $Q''_{4,4,4} \supset Q'_{4,4,4}$ to be generated by $Q'_{4,4,4}$ and $1/2$ times the Kodaira-Néron cycles of the three \tilde{E}_7 's contained in $T_{4,4,4}$. Finally, we define three elements of $Q''_{4,4,4}$. We let f_1 be the element whose multiplicities are given by



and let f_2 and f_3 be the two other elements obtained from f_1 by letting an automorphism of $T_{4,4,4}$ act upon it. We shall need some terminology before the next lemma. We make the obvious notational extension to let a Γ -configuration in a lattice be a collection of elements in a chosen bijection with the vertices of the graph Γ of the lattice all of whose scalar squares are -2 and with scalar products of distinct products of different elements being 1 or 0 according to as the corresponding vertices are connected or not. We note that $e := f_1 + f_2 + f_3$ is the Kodaira-Néron cycle for the \tilde{E}_6 -subconfiguration of $T_{4,4,4}$. We shall also call the tautological $T_{4,4,4}$ -configuration of $Q_{4,4,4}$ (and hence of $Q''_{4,4,4}$) the *standard $T_{4,4,4}$ -configuration* and we shall use the same terminology for subgraphs of $T_{4,4,4}$ and we shall, as for the standard $T_{4,4,4}$ -configuration of curves on a surface of type $T_{3,3,3}$, denote it \mathcal{T} . Note that we shall distinguish between for instance the standard $T_{3,3,4}$ -configuration and the standard $T_{3,4,3}$ -configuration, the first being obtained by removing the degree 1-vertices of the first and second arms and the second by removing those of the first and third arms. Here the arms are numbered so that the up-wards arm is the first, the leftwards arm the second, and the rightwards the third. A $T_{3,3,4}$ -, $T_{3,4,4}$ -, or $T_{4,4,4}$ -configuration that extends the standard $T_{3,3,3}$ -configuration will be said to be *realisable* if all vertices of odd distance to the degree 3 vertex become divisible by 2 in $Q'_{4,4,4}$ (by construction those of distance 1 are already so divisible, thus it is only a condition on those of distance 3).

We need some further notation concerning the standard $T_{4,4,4}$ -configuration \mathcal{T} . For $i = 1, 2, 3$ we let v_i be the degree 1 vertex of \mathcal{T} that appears with multiplicity 1 in f_i and we let e_i the Kodaira-Néron cycle of the \tilde{E}_7 -subconfiguration of the \mathcal{T} that does not contain v_i . We now get a new realisable configuration T by replacing v_i by $e_i - v_i$. It is easily verified that $e_i - v_i = v_i + 2(f_j + f_k)$, where $\{1, 2, 3\} = \{i, j, k\}$, which shows that this new configuration is indeed realisable. Furthermore, it also shows that the lattice spanned by the vertices of T and $1/2$ times the Kodaira-Néron cycles, e'_i , of the \tilde{E}_7 -subconfigurations of T is contained in $Q''_{4,4,4}$.

(as $e'_i = e_i$ and $e'_m = e_m + 2(f_j + f_k)$ for $m = j, k$) and hence comparing discriminants is equal to it. This means that there is a unique isometry, σ_i , of $Q''_{4,4,4}$ that takes the \mathcal{T} to T . We let Γ be the group generated by the σ_i . For a (proper) subset S of $\{1, 2, 3\}$ we let Γ_S be the subgroup generated by the σ_s , $s \in S$. A simple calculation then shows that $\sigma_i(f_i) = 2e - f_i$, $\sigma_i(f_j) = -f_k$, and $\sigma_i(f_k) = -f_j$, where $\{1, 2, 3\} = \{i, j, k\}$. Given a realisable $T_{4,4,4}$ -configuration T in $Q''_{4,4,4}$ and a degree 1 vertex v of it we define a new realisable $T_{4,4,4}$ -configuration obtained from the given one by replacing v with $e' - v$, where e' is the Kodaira-Néron cycle of the \tilde{E}_6 -subconfiguration of T that does not contain v . We shall call this new configuration the *flip* of T with respect to v (or the flip in the i 'th arm if v is the end-vertex of the i 'th arm).

Lemma 4.2 *Let T_1, T_2, \dots, T_n be a sequence of realisable $T_{4,4,4}$ -configurations such that T_{k+1} is a flip of T_k in the i_k 'th arm. Then $T_n = \sigma T_1$, where $\sigma = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n}$.*

PROOF: We prove the result by induction on n and hence we may assume that $T_{n-1} = \tau T_1$, where $\tau = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_{n-1}}$. From this it follows that flipping T_{n-1} in the i_n 'th arm is obtained by applying $\tau \sigma_{i_n} \tau^{-1}$ to it, so that $T_n = \tau \sigma_{i_n} \tau^{-1} T_{n-1} = \tau \sigma_{i_n} T$. \square

Given a realisable $T_{4,4,4}$ -configuration T' we may define vectors v'_i, f'_i , etc by the same formulas as for the standard $T_{4,4,4}$ -configuration \mathcal{T} (in fact, as we shall see, there is a unique isometry of $Q''_{4,4,4}$ taking T to T' and then the vectors corresponding to T' are obtained by applying the isometry to the corresponding elements for T). We shall call the vectors $e \pm f_i$, $i = 1, 2, 3$, the *candidate vectors* and any one of the subsets $\{e \pm f_1\}$ or $\{e - f_1, e - f_2, e - f_3\}$ will be called a *candidate collection*. We shall say that T' is *1-realisable for the i 'th arm* if $\{e' \pm f'_i\} = \{e \pm f_1\}$. If T' is 1-realisable for some i we shall simply say that T' is *1-realisable*. Similarly, if v is a vector that has intersection 0 with all the elements of the standard $T_{3,3,3}$ -configuration except for the end-vertex of the i 'th arm with which it has intersection 1 and furthermore it has intersection 0 with one of $e \pm f_1$ (and then intersection 2 with the other), then we shall say that v is *1-realisable for the i 'th arm*. Similarly, we shall say that v is *2-realisable for the i 'th arm* if it has intersection 2 with exactly one of $\{e - f_1, e - f_2, e - f_3\}$ and intersection 0 with the rest.

Lemma 4.3 i) $Q'_{4,4,4}$ is maximal among integral lattices (i.e., a finitely generated subgroup with integer-valued scalar product) in $Q'_{4,4,4} \otimes \mathbb{Q}$ with the property that $x^2 \equiv C_{4,4,4} \cdot x \pmod{2}$ for all elements x in the lattice.

ii) $Q'_{4,4,4}$ is a unimodular integral lattice.

iii) The f_i span the orthogonal complement of the standard $T_{3,3,3} = \tilde{E}_6$ -configuration and their sum equals the Kodaira-Néron cycle of it. Furthermore, $f_i^2 = -2$ and $f_i \cdot f_j = 1$ for $i \neq j$ and their classes modulo $2Q'_{4,4,4}$ are linearly independent modulo 2.

iv) The candidate vectors are precisely the vectors that are orthogonal to the standard $T_{3,3,3}$ -configuration, have scalar product 0 or 2 with the v_i and have scalar square -2 .

v) The set of extensions of the standard $T_{3,3,3}$ -configuration T to a $T_{3,3,4}$ -configuration (in $Q'_{4,4,4}$) forms a torsor under $T^\perp / \mathbb{Z}E$, where E is the Kodaira-Néron cycle of the standard $T_{3,3,3}$ -configuration: Given a vector v that together with T forms a $T_{3,3,4}$ -configuration and $F \in T^\perp / \mathbb{Z}E$ we associate the vector $v + f - f^2/2E$, where f is the unique vector in T^\perp that is orthogonal also to v and is congruent to F modulo E . Then T and $v + f - f^2/2E$ form a $T_{3,3,4}$ -configuration. It is realisable precisely when F is divisible by 2 in $T^\perp / \mathbb{Z}E$.

vi) The group Γ acts simply transitively on the set of realisable $T_{4,4,4}$ -configurations. The group $\Gamma_{\{1,2\}}$ acts simply transitively on the set of extensions of the standard $T_{3,3,4}$ -configuration to a realisable $T_{4,4,4}$ -configuration (and similarly for the other 2-element subsets of $\{1, 2, 3\}$) and the group $\Gamma_{\{1\}}$ acts simply transitively on the set of extensions of the standard $T_{3,4,4}$ -configuration to a realisable $T_{4,4,4}$ -configuration (and similarly for the other 1-element subsets of $\{1, 2, 3\}$).

vii) Any realisable $T_{4,4,4}$ -configuration can be obtained from any other by a sequence of flips.

viii) Let T be a realisable $T_{4,4,4}$ -configuration which is 1-realisable in the i 'th arm. The flip in the i 'th arm is itself not 1-realisable while the other two flips are. When flipped in any arm but the i 'th the result is 1-realisable in the non-flipped arm which is not 1-realisable in T .

ix) Any element that is 1-realizable for the i 'th arm curve may be obtained from the end-vertex of the i 'th arm of the standard $T_{4,4,4}$ -configuration by 1-realizable flips. In particular, the $T_{3,3,4}$ -, $T_{3,4,3}$ -, or $T_{4,3,3}$ -configuration formed by the element and the standard $T_{3,3,3}$ -configuration can be extended to a 1-realizable $T_{4,4,4}$ -configuration.

x) Any two 1-realizable $T_{4,4,4}$ -configurations can be connected by a sequence of 1-realizable flips.

xi) Any element that is 2-realizable is part of the standard $T_{4,4,4}$ -configuration.

PROOF: We start by splitting off a number of $\mathbf{Z}(-1)$ -factors from $Q'_{4,4,4}$. We can to begin with take the orthogonal complement to all the basis vectors of scalar square -1 . This orthogonal complement has the description one would expect if we were dealing with the dual graph of a configuration of curves. Hence the orthogonal complement is given by the graph D_4 with the square of a non-central vertex being -2 and of the central vertex -1 . Furthermore, $C_{4,4,4}$ projects onto the element in which the central vertex appears with multiplicity 2 and the others with multiplicity 1. We can then consider the orthogonal complement of the central vertex which is described by a configuration of type \tilde{A}_2^* with the difference that the scalar squares of the three basis elements are -1 instead of -2 . This time $C_{4,4,4}$ projects to the element for which the basis elements all appear with multiplicity 1. Finally, we consider the orthogonal complement of one of the basis elements. That complement will then have two basis elements u and v with $u^2 = v^2 = 0$ and $u \cdot v = 2$, i.e., of the form $\mathbf{H}(2)$, where \mathbf{H} is the hyperbolic plane. Furthermore, the projection of $C_{4,4,4}$ is $u + v$. The conclusion is that we have an isometry $Q'_{4,4,4} \cong \mathbf{Z}(-1)^8 \perp \mathbf{H}(2)$ with $C_{4,4,4}$ projecting onto $u + v$ in the last factor. Any integral super-lattice is contained in the lattice dual to $Q'_{4,4,4}$ which by what has just been proven is spanned by $Q'_{4,4,4}$, $u/2$, and $v/2$. As $u/2 \cdot v/2 = 1/2$ any proper super-lattice is spanned by $Q'_{4,4,4}$ and $u/2$, $v/2$, or $(u + v)/2$. However, if x is $u/2$, $v/2$, or $(u + v)/2$ then $x^2 + C_{4,4,4} \cdot x$ is 1, 1, and 3 respectively which in no case is 0 modulo 2.

As for the second part it is easy to show that $Q''_{4,4,4}$ is integral and that $Q_{4,4,4}$ is of index 4 in it so that its discriminant is that of $Q_{4,4,4}$ divided by 4^2 . On the other hand, the discriminant of $Q_{4,4,4}$ is -16 as can be seen for instance from the first part, as we there, implicitly, compute the discriminant of $Q_{4,4,4}(2)$ as $-2^{12} \cdot 2^2$ being of index 2^6 in a lattice of discriminant -2^2 .

As for iii) and iv) they are simple computations.

Turning to v), as $v \cdot E = 1$ we can write any vector in the form $v + f + mE$, where $v \cdot f = 0$. The condition that this vector together with T form a $T_{3,3,4}$ -configuration is equivalent to f being orthogonal to T and $-2 = (v + f + mE)^2 = -2 + f^2 + 2m$, i.e., $m = -f^2/2$. Now the extension is realisable precisely when $v + f - f^2/2E$ is divisible by 2 in $Q'_{4,4,4}$ but as v already is this is equivalent to $f - f^2/2E$ being divisible by 2 in $Q'_{4,4,4}$ but by iii) this in turn is equivalent to $f - f^2/2E$ being divisible by 2 in T^\perp which is also equivalent to F being divisible by 2.

To prove vi) we start by proving that Γ acts transitively on the set of extension of the standard $T_{3,3,3}$ -configuration to a realisable $T_{4,3,3}$ -configuration. An element providing such an extension can, according to v), be written as $v = v_1 + 2f - 2f^2e$ with $v_1 \cdot f = 0$ and it is determined by f modulo $\mathbf{Z}e$. Hence it is enough to show that we may find $\gamma \in \Gamma$ such that $I_\gamma := \gamma v_1 - v_1 \equiv v - v_1 \pmod{\mathbf{Z}e}$ as elements of $T^\perp/\mathbf{Z}e$. Let us start by noting that I_γ fulfills the cocycle condition $I_{\gamma\sigma} = I_\gamma + \gamma I_\sigma$. As Γ fixes T , it also fixes w and induces an action on $T^\perp/\mathbf{Z}e$ and let Γ'' be its image. Furthermore, put $\Gamma' := \Gamma_{\{2,3\}}$. Under the action of Γ on $T^\perp/\mathbf{Z}e$ σ_i induces the reflection in the -2 -element that is the residue of f_i and as $f_1 + f_2 + f_3 = e \equiv 0 \pmod{\mathbf{Z}e}$ it is clear that the natural map from Γ' to Γ'' is surjective. Also as σ_2 and σ_3 fix v_1 we have that $I_\gamma = 0$ for $\gamma \in \Gamma''$. Now by the surjectivity of $\Gamma' \rightarrow \Gamma''$, given $\gamma \in \Gamma$ there is a $\sigma \in \Gamma'$ such that $\gamma' := \gamma\sigma^{-1}$ maps to the identity in Γ'' . This gives $I_\gamma = I_{\gamma'\sigma} = I_{\gamma'} + \gamma' I_\sigma = I_{\gamma'}$ and for another $\gamma_1 \in \Gamma$ we get $I_\gamma + I_{\gamma_1} = I_{\gamma'} + \gamma' I_{\gamma_1} = I_{\gamma'\gamma_1}$ so that the set $S := \{I_\gamma \mid \gamma \in \Gamma\}$ is closed under sums. Similarly, we get $-I_\gamma = I_{\gamma'\sigma^{-1}}$ so that S is a subgroup. Finally, with $\sigma \in \Gamma'$ we have $\sigma I_\gamma = I_\sigma + \sigma I_\gamma = I_{\sigma\gamma}$ so that S is stable under the action of Γ'' . However, $I_{\sigma_1} = -f_1$ and that element generates $T^\perp/\mathbf{Z}e$ as a Γ' -module so that S is indeed equal to $T^\perp/\mathbf{Z}e$.

Hence, to conjugate a given realisable $T_{4,4,4}$ -configuration to the standard one, \mathcal{T} , by an

element of Γ we may assume that the first degree 1 vertex is equal to v_1 . The second degree 2 vertex is then, by v), of the form $v_2 + 2f - 2f^2e$ with $v_2 \cdot f = 0$. The condition that it together with T and v_1 form a $T_{4,4,3}$ -configuration is then that $0 = v_1 \cdot (v_2 + 2f - 2f^2e) = 2(v_1 \cdot f - f^2)$, i.e., $f^2 = v_1 \cdot f$. By iii) f is an integral linear combination of the f_i and the condition that it be orthogonal to v_2 means that it is a linear combination of $g_1 := f_1 + f_2$ and $g_2 := f_2 + f_3$. These two vectors furthermore form a root basis for A_2 . We can represent $v_1 \cdot f$ as $u \cdot f$ where $u = -2/3(g_1 + 2g_2)$ and then, by the Cauchy-Schwartz inequality, $-f^2 = |u \cdot f| \leq \sqrt{-f^2} \sqrt{-u^2} = 2\sqrt{2/3} \sqrt{-f^2}$, i.e., $-f^2 \leq 8/3$. Writing $f = xg_1 + yg_2$ this in turn gives $x^2, y^2 \leq 32/9 < 4$ and hence $|x|, |y| \leq 1$. Going through these possibilities leave us with the solutions $(x, y) = (-1, -1), (-0, 1), (0, 0)$. Now, $\Gamma_{\{2,3\}}$ fixes v_1 and hence permutes these solutions. On the other hand elements of order 3 of $\Gamma_{\{2,3\}}$ fixes only the origin of $T^\perp/\mathbf{Z}e$ and hence each orbit on the solutions has length at least 3 and hence the action is transitive. Hence, we may further assume that the realisable $T_{4,4,4}$ -configuration to be conjugated to \mathcal{T} has its second degree 1 vertex equal to v_2 . The third such vertex has the form $v_3 + 2f - 2f^2e$, the conditions now being that $v_3 \cdot f = 0$ and $v_2 \cdot f = f^2$ and $v_1 \cdot f = f^2$. The two last give $v_1 \cdot f = v_2 \cdot f$ which together with $v_3 \cdot f = 0$ gives that f is a multiple of $g_3 := f_1 + f - 2 + 2f_3$ and writing it as xg_3 makes the remaining condition $v_1 \cdot f = f^2$ equivalent to $-2x^2 = 2x$, i.e., $x = 0, -1$ and hence there are two solutions. As before $\Gamma_{\{3\}}$ acts transitively on these solutions. This shows that Γ acts transitively on the set of realisable $T_{4,4,4}$ -configurations. As the vectors of \mathcal{T} span a lattice of full rank only the identity transformation can fix it. The rest of vi) has actually been proved in the course of the argument.

Turning to vii) we know by vi) that for any two realisable $T_{4,4,4}$ -configurations T and T' that there is a $\gamma \in \Gamma$ such that $T' = \gamma T$ and we shall show that T' can be obtained by a succession of flips by induction on the length of γ (with respect to the generators σ_i , $i = 1, 2, 3$). Hence we may write γ as $\sigma\sigma_i$ and assume that σT can be obtained from T by a succession of flips. Now, $\sigma\sigma_i = \sigma\sigma_i\sigma^{-1}\sigma$ and $\sigma\sigma_i\sigma^{-1}$ takes σT to one of its flips.

As for viii), that T is 1-realisable for the i 'th arm means that $f'_i = \pm f_1$ (where the f'_j are the f 's corresponding to T). Now, a flip in the i 'th arm will take f'_i to $2e - f'_i$ and hence $e' \pm f'_i$ is taken to $3e' - f'_i$ resp. $-e' + f'_i$ neither of which equals $e \pm f_1$. Similarly $e \pm f'_j$ $j \neq i$ is mapped to $e \mp f'_k$, where $\{1, 2, 3\} = \{i, j, k\}$ and they can never equal $e \pm f_1$. This proves that the flip can not be 1-realisable. On the other hand, a flip in the j 'th arm, $i \neq j$, takes $e \pm f'_k$ to $e \mp f'_i$, where $\{1, 2, 3\} = \{i, j, k\}$ and hence the flip is 1-realisable.

Turning to ix), once noted that t^2 maps v_i to $v_i + 2(f_2 - f_3) \bmod \mathbf{Z}e$ and fixes the f_i modulo $\mathbf{Z}e$ the following table for the action of the relevant elements on the v_i is easily established. Note that we display the action on the v_i modulo e and with v_i subtracted.

| | t^{2n} | t^{2n+1} | $t^{2n}\sigma_3$ |
|-------|------------------------|--------------------------|----------------------------|
| v_1 | $2n(f_2 - f_3)$ | $2n(f_2 - f_3) + 2f_2$ | $2n(f_2 - f_3)$ |
| v_2 | $2n(f_2 - f_3)$ | $2n(f_2 - f_3) - 2f_3$ | $2n(f_2 - f_3)$ |
| v_3 | $2n(f_2 - f_3)$ | $2n(f_2 - f_3) - 2f_3$ | $2n(f_2 - f_3) - 2f_3$ |
| | $t^{2n+1}\sigma_3$ | $t^{2n}\sigma_3\sigma_1$ | $t^{2n+1}\sigma_3\sigma_1$ |
| v_1 | $2n(f_2 - f_3) + 2f_2$ | $2n(f_2 - f_3) + 2f_2$ | $2(n+1)(f_2 - f_3)$ |
| v_2 | $2n(f_2 - f_3) - 2f_3$ | $2n(f_2 - f_3)$ | $2n(f_2 - f_3) - 2f_3$ |
| v_3 | $2(n+1)(f_2 - f_3)$ | $2n(f_2 - f_3) - 2f_3$ | $2(n+1)(f_2 - f_3)$ |

(4.4)

Now consider an element v of $Q''_{4,4,4}$ that extends the canonical $T_{3,3,3}$ -configuration to a $T_{4,3,3}$ -, $T_{4,3,3}$ -, or $T_{4,3,3}$ -configuration, in which case we shall say that v *extends the first, second, or third arm respectively*. We shall further assume and also has intersection number 2 with one of $e \pm f_1$ (and then intersection number zero with the other). We want to show that v can be obtained from

By) we can write v as $v_i + 2(f - f^2e)$ with $v_i \cdot f = 0$. Assume first that $i = 1$. Then the intersection with $e \pm f_1$ equals $1 \mp 1 \pm 2f \cdot f_1$. If $v \cdot (e + f_1) = 0$ this gives $f \cdot f_1 = 0$. This implies that $f \equiv n(f_2 - f_3) \bmod \mathbf{Z}e$ and thus that v , according to (4.4), can be obtained from v_1 by (an even number of full round) iterated 1-realisable flips. If instead $v \cdot (e - f_1) = 0$ we get $f \cdot f_1 = -1$

and as a consequence that $f \equiv n(f_2 - f_3) + f_2 \pmod{\mathbf{Z}e}$ so that again it can be obtained from v_1 by (an odd number of full round) iterated 1-realisable flips. (Alternatively, we could replace the standard $T_{4,4,4}$ -configuration by one full round of 1-realisable flips which interchanges the $e \pm f_1$.) The case of $i = 2, 3$ is similar. As 1-realisable flips preserve 1-realisability the last statement of ix) is clear.

To show x) it follows from viii) that we may flip a given 1-realisable configuration so that the result is 1-realisable in the first arm. Now, we have, again from viii), that starting with a flip in the arm to the right resp. left of the 1-realisable arm we may continue flipping by flipping in arms successively in clockwise resp. counter-clockwise fashion keeping 1-realisability. Making a full circle will lead to a configuration which again is 1-realisable in the first arm and the resulting configuration will, by Lemma 4.2 be $t := \sigma_3\sigma_1\sigma_2$ resp. $t^{-1} = \sigma_2\sigma_1\sigma_3$ applied to the original configuration. Now, t acts on v_1, f_1, f_2 , and f_3 by taking them to $v_1 + 2e + 2f_2, -f_1, 4e - f_3$, and $-2e - f_2$ respectively. Now, the end-vertex, v'_1 , of the first arm of the given $T_{4,4,4}$ -configuration is by definition of the form $v_1 + 2f$, where $f \in T^\perp$, T being the standard $T_{3,3,3}$ -configuration. Furthermore, it follows from v) that f is determined by its residue modulo $\mathbf{Z}e$. Doing one full round of flips clockwise changes f to $t(f) + f_2$ modulo $\mathbf{Z}e$ and similarly for counter-clockwise rounds. From this it is easy that by iterating a suitable number of full rounds we may reduce to the case when $f \equiv af_2 \pmod{\mathbf{Z}e}$. We then get that $v'_1 = v_1 + 2(af_2 - (2a^2 - a)e)$ and by v) that $v'_i = v_i + 2(h_i - h_i^2e)$ with $v_i \cdot h_i = 0$ for $i = 2, 3$. The condition that the configuration be 1-realisable in the first arm then means that $2v'_1 - v'_2 - v'_3 = 2v_1 - v_2 - v_3$. That they give a $T_{4,4,4}$ -configuration give the further conditions $v'_i \cdot v'_j = 0$ for $i \neq j$. These conditions translate into a number of linear and quadratic equations and it is easily established that they only have two solutions; $v'_i = v_i$ for $i = 1, 2, 3$ resp. $v'_1 = v_1, v'_2 = v_2 - 2/3(f_2 - f_3)$, and $v'_3 = v_3 + 2/3(f_2 - f_3)$. As the latter does not give solutions in the lattice $Q''_{4,4,4}$ we only have one allowable solution which proves the result.

Finally, to show xi) we note that if v is the end-vertex of the i 'th arm of the standard $T_{4,4,4}$ -configuration, then $v - C$ is orthogonal to all the elements of the standard $T_{3,3,3}$ -configuration and hence is, by iii), a linear combination of the f_i . On the other hand, by assumption and as $(v - C) \cdot e = 0$, we have that $v - C$ is twice the difference of two characteristic functions for elements of the \tilde{A}_2 root basis $\{f_1, f_2, f_3\}$. By [GrLie4-6, Planche I] this is possible only if $v - C$ is a multiple of e , i.e., $v = C + ne$ and taking squares gives $-2 = -2 + 2n$ and that $v = C$. \square

We now use this lemma to determine the Picard groups of an exceptional Enriques surface of type $T_{3,3,3}$ and the normalisation of its canonical cover as well as describe the special 2-sections of its elliptic fibration. If X is an Enriques surface then we shall say that an element of $\text{Num}(X)$ is *effective* if it is the image of an effective divisor. If $v \in \text{Num}(X)$ has square -2 we shall say that it is *given by a -2 -curve* if it is the image of an irreducible curve of self-intersection -2 (in which case the curve is uniquely determined). We start by noticing that an exceptional Enriques surface of type $T_{3,3,3}$ has MW-rank (in the sense given in the introduction) $9 - n$ when the rank of the subgroup of $\text{Num}(X)$ generated by the components of the fibres of the elliptic fibration of X is equal to n .

Proposition 4.5 *Let X be an exceptional Enriques surface of type $T_{3,3,3}$ and let $\tilde{X} \rightarrow X$ be the normalisation of its canonical double cover.*

i) *Any extension of the canonical $T_{3,3,3}$ -configuration to a $T_{4,4,4}$ -configuration of curves is realisable as a configuration of elements of $\text{Num}(X)$. In particular, $\text{Num}(X)$ is identified, using the standard $T_{4,4,4}$ -configuration of curves, with $Q''_{4,4,4}$ and the Picard group of \tilde{X} is identified with $Q'_{4,4,4}$ in such a way that $Q''_{4,4,4}(2) \subset Q_{4,4,4}$ is identified with the image of $\text{Num}(X)$ under the pullback map.*

ii) *A -2 -curve on X is either a component of the elliptic fibration on X or a special 2-section of it.*

iii) *Let v be an element of $\text{Num}(X)$ that extends the canonical $T_{3,3,3}$ -configuration to a $T_{3,3,4}$ -configuration. If it is given by a -2 -curve, its pullback to $\text{Pic}(\tilde{X})$ is divisible by 2. Conversely, if this pullback is divisible by 2, v is effective and an effective divisor of class v is of the form*

$H + V_1 + V_2$, where H is a 2-section of the elliptic fibration, V_1 is a (positive) multiple of the Kodaira-Néron cycle of the \tilde{E}_6 -fibre of the elliptic fibration and V_2 is a sum of components of other fibres.

iv) The set of classes in $\text{Num}(X)$ of extraneous components is either empty (when the MW-rank is 2) or, after possibly renumbering the arms of the canonical $T_{3,3,3}$ -configuration, a candidate collection. It equals $\{e \pm f_1\}$ if the MW-rank is 1 and $\{e - f_1, e - f_2, e - f_3\}$ if it is 0. In particular, when the MW-rank is < 2 then they form a single simple fibre.

v) Let v be an element of $\text{Num}(X)$ that extends the canonical $T_{3,3,3}$ -configuration to a $T_{3,3,4}$ -configuration. If it is given by a -2 -curve then it intersects exactly one extraneous component if such components exist (and then with intersection number 2). Conversely, if v has intersection number 2 with one extraneous component (if extraneous components exist) and intersection number 0 with all the others then it is given by a -2 -curve.

PROOF: That $\text{Num}(X)$ contains $Q_{4,4,4}$ is obvious and it then follows from Lemma 4.1 that it also contains $Q''_{4,4,4}$. By (4.3:ii) $Q''_{4,4,4}$ is unimodular and hence must equal $\text{Num}(X)$. Furthermore, \tilde{X} is a normal rational surface so its Picard group is torsion free and the pullback map gives an injection $Q''_{4,4,4}(2) \subset \text{Pic}(\tilde{X})$. If C is a -2 -curve on X of self-intersection type -1 then as, by Lemma 0.8, its s -invariant is 1, the pullback of it to \tilde{X} is twice a Weil divisor but as its r -invariant is 0 the pullback lies in the smooth part of \tilde{X} and is hence twice a Cartier divisor. Now, the elements of the $T_{4,4,4}$ -configuration the half of which are adjoined to form $Q'_{4,4,4}$ are all represented by such curves C (as the conductrix is as given by Lemma 4.1) and hence $Q'_{4,4,4} \subseteq \text{Pic}(\tilde{X})$. However, by the Riemann-Roch theorem and the fact that the pullback of the conductrix is minus the canonical class we get that $x^2 + C_{4,4,4} \cdot x \equiv 0 \pmod{2}$ for all $x \in \text{Pic}(\tilde{X})$. By (4.3:i) $Q'_{4,4,4}$ is maximal for this property and thus we must have equality.

Turning to ii) assume that C is -2 -curve that is not part of the canonical $T_{3,3,3}$ -configuration. Thus it has non-negative intersection with the conductrix and from Lemma 0.8 we get that the intersection is 0 or 1. When it is 0, C is part of a fibre of the elliptic fibration. When it is 1 it must meet only one curve of the canonical configuration and that curve must appear with multiplicity 1 in the conductrix. On the other hand, again by Lemma 0.8, it can not meet a curve of self intersection type -1 and this only leaves the possibility that it meets one of the end curves of the canonical configuration and then with intersection number 1. This means that it is a 2-section of the elliptic fibration.

As for iii), if v is given by a -2 -curve then that curve must also (for the same reason) have self-intersection type -1 and hence the image of v in $\text{Pic}(\tilde{X})$ must be divisible by 2. Conversely, if the pullback of v to $\text{Pic}(\tilde{X})$ is twice u' , then if u is the norm of u' under the (flat) map $\tilde{X} \rightarrow X$ we have $2u = 2v$ which gives $u = v$ in $\text{Num}(X)$. Hence for the first part of the converse it is enough to prove that u is effective and for that is enough to prove that u' is. Now, as the singularities of \tilde{X} are rational and \tilde{X} is a rational surface we have that $\chi(\mathcal{O}_{\tilde{X}}) = 1$ and the Riemann-Roch formula then gives that $\chi(u') = (u'^2 - k \cdot u')/2 + 1$, where k is the canonical class. Now, the canonical class is minus the pullback of the conductrix giving $u'^2 = -1$ and $k \cdot u' = -1$ and hence $\chi(u') = 1$ so that either u' or $k - u'$ is effective. If $k - u'$ is effective then so is its norm $-2c - u$. Intersecting with e , the \tilde{E}_6 half-fibre of the elliptic fibration, gives $-1 = e \cdot (-2c - u) \geq 0$.

We may write an effective divisor in the numerical class of v as $H + V_1 + V_2$ with H a sum of curves mapping finitely to the base of the elliptic fibration, V_1 a sum of components of the \tilde{E}_6 -fibre, and V_2 a sum of components of other fibres. We have $1 = e \cdot v = e \cdot H$ which forces H to be irreducible and a 2-section. If w_i , $i = 1, 2, 3$, are the three degree 1-vertices of the canonical $T_{3,3,3}$ -configuration \mathcal{C} we have $w_i \cdot v = \delta_{i3}$. If we also have $w_i \cdot H = \delta_{i3}$ then $f \cdot V_1 = 0$ for all vertices of \mathcal{C} and thus H is a multiple of e which is what we want to prove. As $f \cdot H \geq 0$ for all f in \mathcal{C} and as $1 = e \cdot v = e \cdot H$ we get that all but one of the $f \cdot H$ is zero and the exceptional one must appear with multiplicity 1 in e , i.e., be one of the w_i . We are also in the position to be allowed to assume that $w_i \cdot H$ is not δ_{i3} for all i so that we may assume that $w_i \cdot H = \delta_{i2}$. Thus we have that, for f in \mathcal{C} , $f \cdot V_1$ is zero except when f equals w_3 or w_2 and in the latter cases it is 1 resp. -1 . However, as seen from [GrLie4-6, Planche V], that linear form is not represented

by an integral linear combination of elements of \mathcal{C} which gives a contradiction as V_1 is such a combination.

Turning to iv) the case of MW-rank 2 is obvious so we may assume that the MW-rank is < 2 . Let C be an extraneous component. It is disjoint from the elements of the canonical $T_{3,3,3}$ -configuration, is a -2 -curve, and has by Lemma 0.8 intersection 0 or ≥ 2 with any v_i . On the other hand they are part of a fibre of the elliptic fibration and v_i has intersection 1 or 2 with the Kodaira-Néron cycle of such a fibre. This forces the intersection to be ≤ 2 and thus we get only 0 or 2 as possibilities. Hence, by (4.3:iv) (the class of) C is a candidate vector. Furthermore, any two such C 's have non-negative intersection and it is easy to see that a maximal set of candidate vectors whose mutual scalar products are ≥ 0 is of the form $\{e \pm f_i\}$, $i = 1, 2, 3$, $\{e + f_1, e + f_2, e + f_3\}$, or $\{e - f_1, e - f_2, e - f_3\}$. Now the full set of components C 's will sum up to a positive integer multiple of e which means the their classes has to be equal to one of these sets. Hence, the union of the C 's is connected and hence it sums up to e or $2e$ according to as the fibre is double or simple. This shows that the set of classes equals $\{e \pm f_i\}$ or $\{e - f_1, e - f_2, e - f_3\}$ as the number of them plus the MW-rank equals 3 we have proven iv).

Finally, if V is a special 2-section of the elliptic fibration then by (0.8) it intersects any extraneous component with intersection number 0 or ≥ 2 but as V is part of a fibre we get as above only the possibilities 0 or 2. However, as we have just seen in iv), the sum of all the extraneous components equals a simple fibre and thus there will be exactly one V with intersection 2. Conversely, if v is an element extending the canonical $T_{3,3,3}$ -configuration to a $T_{3,3,4}$ -configuration and intersects the extraneous components with intersection number 0 and 2 and exactly one with number 2 if there are extraneous components. By iii) v is numerically equivalent to $H + V_1 + re$, where H is a 2-section, V_1 is a non-negative sum of extraneous components, where we may assume that not all of them appear as their sum is $2e$, and $r \geq 0$. If $V_1 = 0$ we have $-2 = v^2 = H^2 + 2r$ which, as $H^2 \geq -2$ forces $r = 0$ and we are finished. Thus we may assume that $V_1 \neq 0$ and in particular that there are extraneous components and hence that the MW-rank is < 2 . Now, if C is an extraneous component we have $v \cdot C = H \cdot C + V_1 \cdot C$. Hence, if $v \cdot C = H \cdot C$ for then we have $V_1 \cdot C = 0$. This can not, however, be true for all C as that would imply that V_1 is a multiple of e which we have assumed not to be the case. Hence, there must be a C such that $v \cdot C = 0$ and $H \cdot C = 2$ which gives $V_1 \cdot C = -2$.

If the MW-rank equals 1 so that there are only two extraneous components this determines V_1 (as not all extraneous components appear in V_1) and we thus have $V_1 = C$. This gives $-2 = v^2 = H^2 - 2 + 2r + 4$ and as $H^2 \geq -2$ we get a contradiction.

If instead the MW-rank equals 0 then there is an extraneous component D for which $v \cdot D = H \cdot D = 0$ which implies that $V_1 \cdot D = 0$ and then $V_1 = 2C + 2E$, where E is the third extraneous component. By assumption we have $v \cdot E = 2$ and $H \cdot E = 0$ and thus $2 = v \cdot E = 0 + V_1 \cdot E = 0 - 2$, a contradiction. \square

If T is a $T_{4,4,4}$ -configuration of curves on an Enriques surface (which then necessarily is exceptional of type $T_{3,3,3}$) then a flip of the corresponding configuration in $\text{Num}(X)$ will be said to be *effective* if the flip is given by a configuration of curves. Note that we have the following geometric elucidation: Given a $T_{4,4,4}$ -configuration of curves with a distinguished degree 1 vertex v we have a quasi-elliptic fibration with a fibre consisting of the \tilde{E}_6 -subconfiguration which does not involve v . The chosen vertex is part of another reducible fibre of type \tilde{A}_1^* and that fibre is simple precisely when the flip with respect to v is effective. The flip is then obtained by replacing v with the other component of the fibre in which v lies.

Theorem 4.6 *Let X be an exceptional Enriques surface.*

i) *If X is of type $T_{2,3,7}$ then it is \tilde{E}_8 -special in the sense of [CD89, Ch. 3, §4, p. 182]; also there is only one genus 1 fibration on X . The support of the conductrix of X is the union of the curves of the double \tilde{E}_8 -fibre and the curve of cusps of the fibration.*

ii) *If X is of type $T_{2,4,5}$ then it is $\tilde{A}_1 + \tilde{E}_7$ -special in the sense of [CD89, Ch. 3, §5, p. 186] (but see remark below) and there are only two or three genus 1 fibrations on X depending on whether the second reducible fibre of the given fibration is a double or simple fibre. These other*

fibrations have a simple \tilde{E}_8 -fibre and their curve of cusps is the curve of the \tilde{E}_7 -configuration that does not lie on the \tilde{E}_8 -fibre. The support of the conductrix of X is the union of the curves of the double \tilde{E}_8 -fibre and the curve of cusps of the fibration for which it is a fibre.

iii) If X is of type $T_{3,3,3}$ with MW-rank 0 then its elliptic fibration has a simple fibre of type \tilde{A}_2 or \tilde{A}_2^* and it and the double \tilde{E}_6 fibre are the only reducible fibres. It has a unique $T_{4,4,4}$ -configuration (which automatically is extended) and in particular exactly three quasi-elliptic fibrations, each of which has a double \tilde{A}_1^* fibre (and a simple \tilde{E}_7 fibre).

iv) If X is of type $T_{3,3,3}$ with MW-rank 1 then its elliptic fibration has a simple fibre of type \tilde{A}_1 or \tilde{A}_1^* and it and the double \tilde{E}_6 are the only reducible fibres. We may and will number the arms of the standard $T_{4,4,4}$ -configuration such that the two components of the \tilde{A}_2 -fibre have classes $e \pm f_1$ in $\text{Num}(X) = Q''_{4,4,4}$. A $T_{4,4,4}$ -configuration of curves is then 1-realisable and a flip in an arm of such a configuration is 1-realisable precisely when it is effective. Hence any $T_{4,4,4}$ -configuration of curves may be obtained from the standard configuration by a sequence of flips through $T_{4,4,4}$ -configurations of curves.

v) If X is of type $T_{3,3,3}$ with MW-rank 2 fix a $T_{4,4,4}$ -configuration of curves giving an identification of $\text{Num}(X)$ and $Q''_{4,4,4}$. Then every admissible $T_{4,4,4}$ -configuration of elements in $\text{Num}(X)$ is given by a (unique) $T_{4,4,4}$ -configuration of curves. Every extended $T_{2,4,4}$ -configuration of elements that intersects the canonical $T_{3,3,3}$ -configuration in a $T_{2,3,3}$ -configuration is given by a $T_{2,4,4}$ -configuration of curves (and extends to a $T_{4,4,4}$ -configuration of curves). Any flip of a $T_{4,4,4}$ -configuration of curves is effective and any two $T_{4,4,4}$ -configuration of curves are related by a sequence of (effective) flips. In particular there is an infinite number of quasi-elliptic fibrations on X . The \tilde{A}_1^* fibre of a quasi-elliptic fibration is simple.

vi) If X is of type $T_{3,3,3}$ then the set of $T_{4,4,4}$ -configurations of curves on X is a torsor under a group that is the trivial group when the MW-rank is 0, \mathbf{Z} when the MW-rank is 1, and Γ when the MW-rank is 2.

Remark: The definition of $\tilde{A}_1 + \tilde{E}_7$ -special in [CD89, Ch. 3, §5, p. 186] is incorrectly stated; the correct definition is that the two divisors of canonical type (cf. [CD89, III:§1]) should have intersection product 0 and that D_1 should be of type \tilde{A}_1 or \tilde{A}_1^* (this is the condition actually used in subsequent proofs). We shall continue to use the terminology $\tilde{A}_1 + \tilde{E}_7$ -special but now with the indicated modification of meaning.

PROOF: Assume that X is exceptional of type $T_{2,3,7}$. As the \tilde{E}_8 -fibre is double, the curve of cusps intersects the \tilde{E}_8 -configuration in the last vertex of the long arm by (2.2) and so the fibre together with the curve of cusps form a $T_{2,3,7}$ -configuration. By definition this is nothing but the configuration necessary for \tilde{E}_8 -speciality and then by the proof of [CD89, Prop. 3.4.1] there is only one genus 1 fibration on X . The support of the conductrix can be read off from the tables of (2.2).

Assume now that X is exceptional of type $T_{2,4,5}$. As the \tilde{E}_7 -fibre is double, the curve of cusps intersects the \tilde{E}_7 -configuration in the last vertex of a long arm by (2.2). Furthermore, as the sum of the number of components of irreducible fibres minus the number of such components is equal to 8 for a quasi-elliptic fibration there is one other reducible fibre which is necessarily a \tilde{A}_1^* -fibre. Depending on whether or not that fibre is double or simple the intersection number of the curve of cusps and it is 1 or 2. In any case the curve of cusps and these two reducible fibres form the configuration required for $\tilde{A}_1 + \tilde{E}_7$ -specialty. The statement on the genus 1 pencils follows from the proof of [CD89, Prop. 3.5.2] and the support of the conductrix can be read off from the tables of (2.2).

Assume now that X is of type $T_{3,3,3}$ and of MW-rank 0. By (4.5:iv) the extraneous components have classes $\{e - f_1, e - f_2, e - f_3\}$ in $\text{Num}(X)$ and thus form a \tilde{A}_2 - or \tilde{A}_2^* -fibre F . Consider now a -2 -curve C on X that is not in a fibre of the elliptic fibration. By (4.5:ii) it is then a 2-section and hence intersects the canonical $T_{3,3,3}$ -configuration in some end vertex. By Lemma 0.8 C will have intersection number 0 or ≥ 2 with each extraneous component and as these intersection numbers add up to 2 we get exactly one with intersection 2. Hence we may apply

(4.3:xi) to conclude that C is part of the standard $T_{4,4,4}$ -configuration. In particular there is only one $T_{4,4,4}$ -configuration of curves. The rest of iii) then follows from Lemma 4.1.

Assume now that X of type $T_{3,3,3}$ and of MW-rank 1. This time we get two extraneous components of classes $e \pm f_1$ giving a fibre of type \tilde{A}_1 or \tilde{A}_1^* . Using this time (4.3:x) together with (4.5:v) proves iv).

Assume now that X is of type $T_{3,3,3}$ with MW-rank 2. Consider a realisable $T_{4,4,4}$ -configuration T' . An end-vertex v'_i is, by (4.5:v), represented by a -2 -curve. This proves v).

Finally turning to vi), when X is of type $T_{3,3,3}$ and of MW-rank 0, then there is a single $T_{4,4,4}$ -configuration of curves which is a torsor under the trivial group and when X has MW-rank 2 Γ acts transitively and simply through flips on the set of $T_{4,4,4}$ -configurations of curves. For the case of MW-rank 1 we note that we have just proved that for a given $T_{4,4,4}$ -configuration of curves there are exactly two effective flips of it and if the configuration is already the effective flip of a configuration then one of the effective flips is the inverse of the performed flip. This means that we get an action of \mathbf{Z} on configurations by letting 1 act by flipping in the arm to the left (say) of the arm in which it is 1-realisable and -1 by flipping in the arm to the right. By v) this is a transitive action and it is easy to see that it is simple. \square

We are now prepared to describe all the -2 -curves on an exceptional Enriques surface.

Theorem 4.7 *Let X be an exceptional Enriques surface.*

- i) *If X is of type $T_{2,3,7}$ then all -2 -curves on X are part of the conductrix.*
- ii) *If X is of type $T_{2,4,5}$ then all -2 -curves on X are either part of the conductor or part of a fibre of the quasi-elliptic pencil with half-fibre the (unique) \tilde{E}_7 -subconfiguration of the conductrix.*
- iii) *If X is of type $T_{3,3,3}$ then all -2 -curves on X are either part of the fibres of the elliptic fibration on X or are one of the end-vertices of a $T_{4,4,4}$ -configuration of curves on X . In the latter case it is a member of at most 6 such configurations. Furthermore, such a curve is obtained from one of the end-vertices of the standard $T_{4,4,4}$ -configuration by a sequence of effective flips. In particular there are an infinite number of them precisely when the MW-rank is < 2 .*

PROOF: The first part follows from [CD89, Prop. 2.4.4] combined with the fact that the $T_{2,3,7}$ root basis is crystallographic ([loc. cit., Prop. 2.4.3]).

As for the second part we add to the conductrix $T_{2,4,5}$ -configuration the components of the other reducible fibres of the quasi-elliptic fibration with a simple \tilde{E}_7 -fibre. This gives us one of two possible root bases which are given by the graphs that are at the bottom of [loc. cit., p. 187] and at the top of [loc. cit., p. 188] respectively and they both are crystallographic as is proven in [loc. cit., p. 188].

Turning to the last part, a -2 -curve is either in a fibre of the elliptic fibration or a 2-section (cf. (4.5:ii)). By Lemma 4.3 the curve is part of a realisable, 1-realisable, resp. 2-realisable $T_{4,4,4}$ -configuration and by (4.5:v) such a configuration is configuration of curves. \square

We can now finish this section by proving Theorem C.

Proof of Theorem C: The theorem follows from Theorem 4.6 together with Lemma 4.1 in the $T_{3,3,3}$ -case to show that every quasi-elliptic fibration comes from a $T_{4,4,4}$ -configuration of curves except for the identification of Γ as a Coxeter group. However, by construction Γ fixes the standard $T_{3,3,3}$ -configuration and hence acts faithfully on the orthogonal complement of it. It is then easy to see that as such it acts as the \tilde{A}_2 Coxeter group (with root basis the $f_i - e/2$). \square

5 Classification of exceptional Enriques surfaces

As we have seen, the exceptional Enriques are exactly the surfaces admitting a special genus 1 fibration with a double fibre of type \tilde{E}_n , $n = 6, 7, 8$. Such fibrations have been classified in [Sa03] and for the reader's convenience we repeat that classification here. All of them are presented as minimal resolutions of a double cover of the ruled surface $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(2)_s \oplus \mathcal{O}_{\mathbf{P}^1}(t))$ and the projection to the base gives the genus 1 fibration. We shall use x and y as homogeneous

coordinates on \mathbf{P}^1 and z as the variable for the double cover. We make the double cover with respect to $\mathcal{O}_{\mathbf{P}}(2) \otimes \mathcal{O}_{\mathbf{P}^1}(1)$ so that the double cover has equation $z^2 + kz + \ell = 0$ with $k \in \Gamma(\mathcal{O}_{\mathbf{P}}(2) \otimes \mathcal{O}_{\mathbf{P}^1}(1)) = \Gamma(S^2(\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)) \otimes \mathcal{O}_{\mathbf{P}^1}(1))$ and $\ell \in \Gamma(\mathcal{O}_{\mathbf{P}}(4) \otimes \mathcal{O}_{\mathbf{P}^1}(2))$. In the case of $\mathbf{Z}/2$ -surfaces (or equivalently when the genus 1 fibration has two double fibres) we have the following equations:

- \tilde{E}_6 : $z^2 + (bx^3y^2s^2 + v^2x^2yst)z + (y^4 + x^4)x^3y^3s^4 + (v^3xy + ax^2)x^3y^3s^3t + v^2x^3y^3s^2t^2 + xyt^4 = 0$ where a and b are arbitrary constants and $v \neq 0$. It is also easily verified¹ that for $(a, b, v) = (1, 1, 1)$ it has MW-rank 2, for $(a, b, v) = (1, 0, 1)$ it has MW-rank 1, and for $(a, b, v) = (1, \zeta, 1)$, where ζ is a fifth root of unity, it has MW-rank 0 (in the last two cases the singularity that when resolved gives an extra reducible fibre are at $(x, y, s, t, z) = (1, 1, 1, 0, 0)$ and $(x, y, s, t, z) = (1, \zeta^3 + 1, 1, \zeta^3 + \zeta^2 + 1, \zeta^2 + \zeta + 1)$ respectively).
- \tilde{E}_7 : $z^2 + bx^3y^2s^2z + (y^4 + x^4)x^3y^3s^4 + ax^5y^3s^3t + xyt^4 = 0$ where a is arbitrary and $b \neq 0$.
- \tilde{E}_8 : $z^2 + (y^4 + x^4)x^3y^3s^4 + ax^5y^3s^3t + xyt^4 = 0$ where $a \neq 0$.

In the case of α_2 -surfaces (i.e., fibrations with one double fibre) we have the following equations:

- \tilde{E}_6 : $z^2 + (xy + bx^2)x^3s^2z + x^3y^7s^4 + (y^3 + ax^3)x^5s^3t + x^4st^3 + xyt^4 = 0$ with a and b arbitrary.
- \tilde{E}_7 : $z^2 + x^5s^2z + x^3y^7s^4 + ax^8s^3t + xyt^4 = 0$ with a arbitrary.
- \tilde{E}_8 : $z^2 + x^3y^7s^4 + x^8s^3t + xyt^4 = 0$.

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¹A Magma file for performing these calculations is to be found as <http://www.math.su.se/~teke/excenr.mg>.